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A pseudo-sound constitutive relationship for the dilatational covariances in compressible turbulence

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The mathematical consequences of a few simple scaling assumptions regarding the effects of compressibility are explored using a singular perturbation idea and the methods of statistical fluid mechanics. Representations for the pressure–dilatation and dilatational dissipation appearing in single-point moment closures for compressible turbulence are obtained. The results obtained, in as much as they come from the same underlying procedure, represent a unified development for both dilatational covariances. While the results are expressed in the context of a statistical turbulence closure they provide, with very few phenomenological assumptions, an interesting and clear mathematical model for the 'scalar' effects of compressibility. For homogeneous turbulence with quasi-normal large scales the expressions derived are – in the small turbulent Mach number *squared* isotropic limit – exact. The expressions obtained contain constants that have a precise physical significance and are defined in terms of integrals of the longitudinal velocity correlation. The pressure–dilatation covariance is found to be a non-equilibrium phenomenon related to the time rate of change of the kinetic energy and internal energy of the turbulence; it is seen to scale with $\alpha^2 M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon_s)^2$. Implicit in the scaling is a dependence on the square of a gradient Mach number, $S\ell/c$. A new feature indicated by the analysis is the appearance of the Kolmogorov scaling coefficient, α, suggesting that large-scale quantities embodied in the well-established $\varepsilon \sim \tilde{u}^3/\ell$ relationship provide a link to the structural dependence of the effects of compressibility. The expressions for the dilatational dissipation are found to depend on the turbulent Reynolds number and scale as $M_t^4 (Sk/\varepsilon_s)^4 R_t^{-1}$. The scalings for the pressure–dilatation are found to produce an excellent collapse of the pressure–dilatation data from direct numerical simulation.

1. Introduction

Compressible shear layers are encountered in many practical applications ranging from supersonic injectors for mixing, to entrainment in gas turbines and to scramjet combustion in hypersonic vehicles. They play a role in the generation of noise from jet and rocket engines. There are several features of these shear layers important to the mixing problem that cannot be predicted with current computational models. The compressible mixing layer has a growth rate much lower than its incompressible counterpart (Papamoschou & Roshko 1988; Bradshaw 1977). The effects of compressibility also contribute to substantial reductions in turbulence levels (Elliott & Samimy 1991), and reductions of the turbulent shear stress (Blaisdell, Mansour & Reynolds

1991; Sarkar 1995) while increasing levels of normal stress anisotropy (Goebel & Dutton 1991; Elliot, Samimy & Arnette 1995; Vreman, Sandham & Luo 1996; and Simone, Coleman & Cambon 1997). The reviews by Lele (1994), Gutmark, Schadow & Yu (1995), and Spina, Smits & Robinson (1994) expand on this subject.

An analytical development appropriate to high-Reynolds-number and high Mach number transversely sheared flows with small bulk dilatation and low M_t^2 are the subject of this article. These restrictions are satisfied in a large number of transversely sheared flows ranging from simple shear layers of theoretical interest (Papamoschou & Roshko 1988), to complex shear layers found in supersonic mixing enhancement (Gutmark *et al.* 1995). Such flows, important in mixing enhancement and noise reduction, are associated with the different nozzle shapes or flow configurations such as multiple jets, coaxial jets, countercurrent mixing layers, ramped nozzles, normal/tangential injection or vortex generators. In these supersonic shear layers a Mach number based on the fluctuating velocity is small: a Mach 4 mean flow, for example, with a turbulence intensity of 8% has a turbulent Mach number of $M_t = 0.32$. The square of this turbulent Mach number, the appropriate perturbation expansion parameter arising from the Navier–Stokes equations, $M_t^2 = 0.1$, is small and allows a perturbative approach with analytical results.

In the context of single-point two-equation turbulence closure methods, compressibility effects due to the fluctuating divergence explicitly appear in two terms in the kinetic energy equation: the pressure–dilatation, $\langle pd \rangle$, and the variance of the dilatation, $\langle dd \rangle$, which is related to what has come to be called the compressible dissipation, $\varepsilon_c = \frac{4}{3}\nu\langle dd \rangle$. The pressure–dilatation and the dilatational dissipation appear in the turbulent energy equation

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\, k = \bar{\rho}P_k - \bar{\rho}\varepsilon_s + \langle pd \rangle - \bar{\rho}\varepsilon_c + T_k \tag{1}$$

and, with opposite sign, in the internal energy equation,

$$\bar{\rho}c_v\frac{\mathrm{D}}{\mathrm{D}t}\, T = P_T - \langle pd \rangle + \bar{\rho}\varepsilon_s + \bar{\rho}\varepsilon_c + T_T, \tag{2}$$

here written in terms of the mean temperature with a constant c_v. The dilatational covariances represent an irreversible, ε_c, and reversible transfer, $\langle pd \rangle$, of energy between the mean internal energy field and the fluctuating kinetic energy field as shown in figure 1. (A similar figure is given in Huang, Coleman & Bradshaw 1995.) Interchanges of energy between the mean kinetic energy and mean temperature, involving the mean pressure–dilatation and viscous heating do not require closure. P_k represents the production, $-\{v_i v_j\}V_{i,j}$, and the T_k and T_T represent transport terms. Upper-case letters represent mean values while lower-case letters denote fluctuating values except in the case of the mean density, $\bar{\rho}$. The angle brackets represent a Reynolds average and the curly brackets indicate a Favre average. In addition to the articles already mentioned, additional insight into these quantities can be found in the studies of Durbin & Zeman (1992), Zeman & Coleman (1993), Zeman (1993), Erlebacher *et al.* (1990), Sarkar (1992), Sarkar, Erlebacher & Hussaini (1991*a*), Sarkar *et al*, (1991*b*), Blaisdell *et al.* (1991), Blaisdell & Sarkar (1993).

It is useful to give a simplified overview of what is currently believed, in the context of a *statistical closure*, to be the nature of the effects of compressibility. In this context the instantaneous dynamics are averaged and their net cumulative effect is found in several different terms in the moment equations. The net effects of compressibility, as currently understood and as appearing in the energy budget

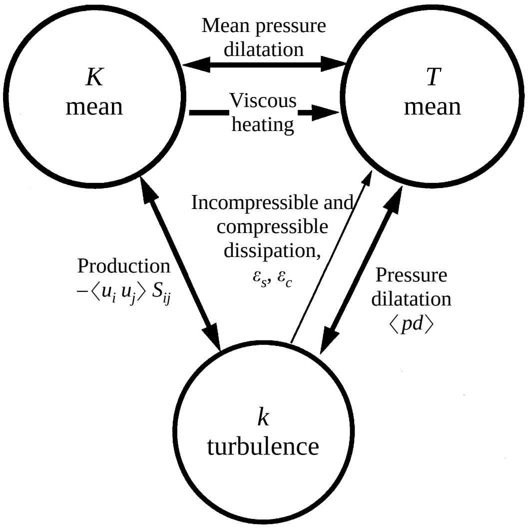

FIGURE 1. Energy transfers in compressible turbulence.

are manifest through: (i) the dilatational dissipation, ε_c, (ii) the pressure–dilatation covariance, $\langle pd \rangle$, (iii) reductions in shear anisotropy, b_{12} accompanied by increases in the streamwise the anisotropy, b_{11} and decrease in the transverse anisotropy, b_{22}, and (iv) changes in the relative strain rates, Sk/ε, characterizing a compressible flow. Here $b_{ij} = \{v_i v_j\}/2k - \frac{1}{3}\delta_{ij}$ is the anisotropy tensor. For the homogeneous flows there is very little difference between a Favre average and a Reynolds average (indicated by the angle brackets). These several effects influence the growth rate of k. Defining a non-dimensional growth rate of k for homogeneous shear turbulence as $\lambda = \dot{k}/Sk$ and rearranging the $\dot{k}$ equation for a homogeneous shear produces

$$\lambda = \frac{\dot{k}}{Sk} = -2b_{12} - \frac{\varepsilon}{Sk} + \frac{\langle pd \rangle}{Sk} - \frac{\varepsilon_c}{Sk} = -\lambda_b - \lambda_\varepsilon - \lambda_{pd} - \lambda_{\varepsilon_c}. \quad (3)$$

In the equation the overdot represents a time derivative. The importance of the effects of compressibility in direct numerical simulations (DNS) of homogeneous shear can be ordered, approximately, as $\lambda_{\varepsilon_c} < \lambda_{pd} < \lambda_b$. Too little is known about the effects of compressibility on λ_ε to say anything definitive. This article investigates λ_{pd} and λ_{ε_c} due to the scalars $\langle pd \rangle$ and $\langle dd \rangle$ appearing in the energy budget; one might call these the *scalar* aspects of the effect of compressibility. This has also been called, quite sensibly, an 'energetic' approach to the effects of compressibility (Simone *et al.* 1997).

Dilatational dissipation

Work by Zeman (1990) began by looking at the dilatational dissipation. It was expected to be an important feature of compressible turbulence in hypersonic flows in which it appears as an additional dissipative mechanism in the turbulence energy equation. Since Zeman's early contributions, work has focused on more modest supersonic flows; the dilatational dissipation, once thought to be responsible for the full reduction of the mixing in the mixing layer (Sarkar *et al.* 1991*a,b*; Sarkar & Lakshmanan 1991) now appears to be a small portion of the stabilizing effects as indicated in homogeneous shear DNS (Blaisdell *et al.* 1991; Sarkar 1995), for this class of flows. This is consistent with more recent results of Vreman *et al.* (1996) and Simone *et al.* (1997).

Pressure–dilatation

Sarkar (1992) has since found that, for homogeneous shear DNS, the pressure–dilatation covariance is more important than the dilatational dissipation. This is consistent with the present analytical results – the dilatational dissipation is found to scale with M_t^4 and inversely with the Reynolds number. The present results also indicate that the pressure–dilatation scales with the magnitude of the departure from equilibrium. It is for this reason that Vreman *et al.* (1996) reports a negligible pressure–dilatation: the flow that is the subject their observations is *stationary* (Ristorcelli 1995). Of the many fully turbulent simple flows the homogeneous shear is arguably the one with the largest departure from equilibrium and the pressure–dilatation is not negligible. The importance of the pressure–dilatation will depend on the flow.

The dilatational effects investigated in this article are scalar and reduce the Reynolds stress through the reduction of k in the evolution equation for k through the agencies of $\langle pd \rangle$ and ε_c. These dilatational covariances are not the most important effects of compressibility, and do not explain the substantial decrease of k seen in select flows as discussed below. They are also not, in the homogeneous shear, negligible effects: in the DNS of Blaisdell (1996, personal communication) they are seen to be of the order of 5–10% of the dissipation.

Structural anisotropy

In homogeneous shear DNS, Sarkar (1995) has studied the effects of compressibility on the 'structural' anisotropy as indicated by the anisotropy tensor $b_{ij} = \{v_i v_j\}/2k - \frac{1}{3}\delta_{ij}$. His results and arguments suggests that the stabilizing effects of compressibility are associated with a decrease in b_{12} and this can be related to an increase in the gradient Mach number, $S\ell/c$. It is possible to see a similar reduction in b_{12} in the inhomogeneous shears of Elliott & Samimy (1990), Goebel & Dutton (1991), Elliot *et al.* (1995). A reduction in b_{12} is also seen in the homogeneous DNS of Blaisdell *et al.* (1991) where it appears that a combination of increasing compressibility and larger relative strain results in anisotropies notably different from that found in incompressible flows (Abid 1994; Speziale, Abid & Mansour 1995). The decrease in k appears to be linked to the decrease in the production of the turbulent shear stress, $P_{12} \sim -(b_{22} + \frac{1}{3})V_{1,2}$ due to a sizeable decrease in b_{22}. These structural effects of compressibility, as described in Vreman *et al.* (1996) and Simone *et al.* (1997), are larger and are not addressed. The structural effects on the Reynolds stress, $\{uv\} = 2b_{12}k$, occurring through the reduction of b_{22} and b_{12} will require a compressible representation of the pressure–strain covariance. The thesis by Adumitroaie (1997) offers one possible procedure to this end.

The present objective is to investigate the consequences of a pertubative approach for the effects of compressibility in an investigation of $\langle pd \rangle$ and ε_c. This will provide (i) an asymptotic benchmark to which fully nonlinear developments can be compared, (ii) delineate analytically the nondimensional parameters and scalings that play a roll in compressible flows, (iii) lay-out a procedure from which to obtain representations for the more difficult effects associated with the pressure–strain covariance and (iv) take a preliminary step towards a computational model for a portion of the effects of compressibility relevant to a sizeable class of high-speed aeronautical flows.

This article is primarily analytical. While numerical validation of the analysis is provided space and subject matter indicate that this is more satisfactorily addressed in an article devoted to the details of turbulence modeling and validation issues (Ristorcelli 1997). The results presented herein are a rigorous mathematical consequence of a few very reasonable assumptions. The present approach is to be contrasted to other analytical developments, for example, Durbin & Zeman (1992), Cambon, Coleman & Mansour (1993) or Simone *et al.* (1997) which treat the linear rapid distortion problem in which the mean rate of strain is much higher than the self-straining by the turbulence. The present approach treats the nonlinear problem; the fluctuating strain, associated with the nonlinear turbulence interactions over the range of scales that occur at a high Reynolds number, is large in comparison to the mean strain or rotation. This includes the situation in which nonlinear effects have had time to respond to changes in mean gradients. This methodology is expected to be applicable to a variety of tranversely sheared flows as might occur in situations where the control of mixing or acoustics is important. This methodology will not apply to RDT flows such as those treated Durbin & Zeman (1992). The intent is to produce analytical models for some effects of compressibility providing some insight into the nature of the effects of compressibility in the nonlinear regime. The intent is also to describe and validate a rational and systematic procedure for understanding the effects of compressibility for compact (see below) low fluctuating Mach number flows of engineering interest. That the more important pressure–strain effect is not first treated is due to the complexity of that modelling using a pseudo-sound approach – a procedure that first requires validation before such a substantial effort can proceed with any certainty.

The present approach for the effects of the fluctuating dilatation differs from the approaches of Zeman or Sarkar. In short: a low turbulent Mach number expansion of the equation of state, the Navier–Stokes, the continuity and wave equations is conducted. At low M_t the problem is recognized as having two relevant length scales an inner scale: ℓ, associated with the turbulence field, and an outer scale $\lambda \sim \ell/M_t$ associated with an acoustic radiation field surrounding and generated by the vortical motion. The perturbation development produces an *algebraic* constitutive equation, on the inner scale, for the fluctuating dilatation. Taking the relevant moments of the expression produces constitutive relations for $\langle pd \rangle$ and $\langle dd \rangle$. Assuming homogeneity and quasi-normality, expressions without any *undefined* constants are obtained for $\langle pd \rangle$ and $\langle dd \rangle$.

This article is organized as follows: governing equations, analysis, discussion of implications of the results and delineation of the limitations and assumptions. Section 2 gives a simple heuristic picture of the physics after which equations consistent with the physics are derived. In §3 the assumptions of homogeneity and quasi-normality are exploited in obtaining leading-order analytical expressions for the desired covariances. The methods of statistical fluid mechanics, following the inceptional works of von Kármán & Howarth (1938), Batchelor (1951), Kraichnan (1956), and Proudman (1952), are relied on extensively. As a byproduct of the subsection on the rapid pressure–dilatation correlation a leading-order expression for the pressure-variance in an arbitrary three-dimensional mean flow is derived – this is a generalization of Kraichnan's (1953) result. This is followed in §4 by a description, in the context of a few simple flows, of the consequences of the results; issues such as the Reynolds-number scaling of the compressible dissipation and the gradient Mach number are discussed. In §5 the justification and limitations of the assumptions made in the analytical section, which may appear to have been given short shrift in earlier sections, are described in greater detail.

2. Governing equations

It is useful to keep in mind one essential and central piece of physics that forms the lynchpin of the pseudo-sound theory and makes the present method and results possible: in the *near field* of a compact acoustic source the fluid behaves as if it were incompressible. This observation appears to have been first made by Landau & Lifshitz (1985) and is a cornerstone in the method of matched asymptotic expansions in acoustics. This is a consequence of the fact that the Laplacian operator in the wave equation, when scaled on the time and length scales of the turbulence, is the most singular operator. This fact is recognized in acoustics in problems involving compact sources, low-frequency motions, or acoustic fields near singularities.

A few ideas form the foundation of the present pseudo-sound analysis for the dilatational covariances:

Compact source

It is assumed that the turbulence produces the pressure and density fluctuations in the medium; the frequencies of the compressible disturbances are then the same as the frequencies of the turbulence, $c/\lambda \sim \tilde{u}/\ell$. The characteristic velocity fluctuation scale is indicated by $\tilde{u}$. The problem is a singular perturbation problem: there is a correlation length scale associated with the fluctuations of the turbulence, ℓ, and a length scale $\lambda \sim \ell/M_t$ associated with the propagation of pressure and density fluctuations. Here $M_t = (2k/3)^{1/2}/c$ is the turbulent Mach number where $k = \frac{1}{2}\langle u_j u_j \rangle$ and $c_\infty^2 = \gamma P_\infty/\rho_\infty$ is the sound speed. Associated with the low-M_t assumption which leads to the two disparate scales is what is called, in aeroacoustics, the *compact-source* assumption: a turbulent eddy is small with respect to the length scale of its acoustic radiation. Closely related to these two length scales are two time scales: one associated with the convective modes of the flow, say $\tau_t \sim \ell/\tilde{u}$, and the other with the sound crossing time, $\tau_c \sim \ell/c$ – the time it takes for information to cross a typical length scale of the turbulence. Note that $\tau_t/\tau_c \sim M_t$.

Convective and propagating pressures

Two pressures, an 'acoustic' pressure which propagates and a 'pseudo-pressure' associated with the convective motions of the fluid, are distinguished. The term 'pseudo-pressure' was first coined by Blokhintsev (1956) as quoted in Ribner (1962). The pressure fluctuations in a fluid satisfy, within the adiabatic approximation, the following wave equation (Lighthill 1952):

$$c_\infty^{-2} p_{,tt} - p_{,jj} = \Box^2 p = (\rho u_i u_j)_{,ij} \tag{4}$$

where p represents the deviations of the fluid pressure from its reference value. Solutions to this equation are composed of the homogeneous solution, which obeys the sourceless wave equation, $\Box^2 p = 0$, and the wave equation with source due to turbulent fluctuations. Following Ribner (1962) the fluid pressure is decomposed into convective and propagating parts $p = p_c + p_p$ where p_c satisfies $p_{c,jj} = -(\rho u_i u_j)_{,ij}$ and therefore p_p satisfies

$$c_\infty^{-2} p_{p,tt} - p_{p,jj} = -c_\infty^{-2} p_{c,tt}. \tag{5}$$

In the region of the fluid turbulence the pseudo-pressure is much larger than the propagating pressure. Far from the vortical portions of the motion, the propagating pressure is the major portion of the pressure field. The pseudo-pressure decays quite rapidly (Ribner 1962). Thus there is an inner region of scale ℓ in which the major portion of the pressure is associated with the vortical motions and an outer region, in which the much smaller propagating acoustic pressure is the major component of the pressure field. In the inner region of scale $\ell \ll \lambda$, the sound speed is effectively infinite: on a time scale of the flow, signals are felt throughout the region of scale ℓ effectively simultaneously. To obtain the dilatational covariances with the dilatation only the *inner* solution of the singular perturbation problem, in which the pseudo-pressure dominates, is required.

Compact flow

The use of the inner solution is a useful approximation in mediums that are finite or infinite in extent *for covariances involving at least one fluctuating quantity that does not propagate*. Contributions to the covariances from regions outside of the correlation length, the outer solution, are negligible. This point can be made by decomposing pressure and dilatational field into portions associated with the local eddy, (p^ℓ, d^ℓ), and those whose source is the vortical motions an integral length scale or more away, (p^u, d^u). The pressure–dilatation can then be written

$$\langle pd \rangle = \langle p^\ell d^\ell \rangle + \langle p^\ell d^u \rangle + \langle p^u d^\ell \rangle + \langle p^u d^u \rangle. \tag{6}$$

The middle two terms, as they arise from portions of the flow that are not correlated, are zero. The quantity $\langle p^u d^u \rangle$ is not zero; it is, however, of order M_t^2, with respect to $\langle p^\ell d^\ell \rangle$ since p^u in the local eddy is the acoustic pressure associated with a distant (uncorrelated) eddy. Thus

$$\langle pd \rangle = \langle p^\ell d^\ell \rangle + O(M_t^2). \tag{7}$$

This is not the case for the variance $\langle dd \rangle$ as d is associated with a propagating field. A similar argument produces

$$\langle dd \rangle = \langle d^\ell d^\ell \rangle + \langle d^\ell d^u \rangle + \langle d^u d^\ell \rangle + \langle d^u d^u \rangle = \langle d^\ell d^\ell \rangle + \langle d^u d^u \rangle. \tag{8}$$

The present local theory does not and cannot account for non-local contributions represented by $\langle d^u d^u \rangle$. It is for this reason that, in this article, the dilatational dissipation will often be referred to as the 'local dilatational' dissipation as it is assumed that $\langle dd \rangle = \langle d^\ell d^\ell \rangle$. In the compact flow assumption $\langle d^u d^u \rangle$ makes very little contribution. These $\langle d^u d^u \rangle$ terms are expected to be important in compressible homogeneous DNS in which the flow domain is effectively infinite.

This is not the case for compact flows. Most flows of engineering interest are compact flows. The *compact flow* assumption is a statement that the size of the turbulent field, D, is small or on the order of the acoustic scale, $D/\lambda \leqslant 1$. This requires a low M_t; consider that $D \sim \ell$ and that $\lambda \sim \ell/M_t$, then $D < \lambda$ and the flow is compact. As most engineering flows, at low M_t, being only few integral scales wide are compact flows, the contributions from $\langle d^u d^u \rangle$ originating more than an integral scale away, are expected to be negligible. In compact flows the primary contribution to $\langle dd \rangle$ is $\langle d^\ell d^\ell \rangle$ for which the present analysis is valid.

Uniform convergence

It is required that the equations should, with bounded derivatives, uniformly approach their incompressible form as the Mach number goes to zero. These facts are used to produce the gauge functions in a perturbation expansion in which the small parameter is related to the Mach number of the velocity fluctuations.

The governing equations

The following equations are used to describe the portion of the flow of interest:

$$\rho_{,t} + u_k \rho_{,k} = -\rho u_{p,p}, \tag{9}$$

$$\rho u_{i,t} + \rho u_k u_{i,k} + p_{,i} = 0, \tag{10}$$

$$p/p_\infty = (\rho/\rho_\infty)^\gamma. \tag{11}$$

The momentum and continuity equations can be combined to give the following equation:

$$\rho_{,tt} - p_{,jj} = (\rho u_i u_j)_{,ij} \tag{12}$$

which becomes a wave equation for ρ or p if the gas law is used to eliminate one in favour of the other. For clarity of exposition the viscous terms are not carried: they can be shown to be of higher order for the compressible portions of the field, see for example Zank & Matthaeus (1991). Since a spectral Mach number exhibits an approximate wavenumber dependence of $\kappa^{-1/3}$ the scales of the motion which contain fluctuating dilatation modes will be, for a high-Reynolds-number flow, larger scales of the motion than those most influenced by viscosity.

Perturbing about a reference state, (p_∞, ρ_∞), the non-dimensional forms of the pressure and density are taken as $p = p_\infty(1 + p')$, $\rho = \rho_\infty(1 + \rho')$. The independent variables are rescaled with the energy-containing length and time scales of the 'inner' fluctuating field: $\ell/\tilde{u}$ and ℓ. (Here $\tilde{u}$ represents a characteristic scale for the fluctuating velocity.) There will be no need for the outer expansion; our interest is in the effects of compressibility on the vortical modes of the flow and not the acoustic propagation problem. Dropping primes the equations become

$$\rho_{,t} + u_p \rho_{,p} = -(1+\rho) u_{p,p}, \tag{13}$$

$$(1+\rho) u_{i,t} + (1+\rho) u_p u_{i,p} + \epsilon^{-2} p_{,i} = 0, \tag{14}$$

$$p - \gamma\rho = \tfrac{1}{2}\gamma(\gamma-1)\rho^2, \tag{15}$$

$$\rho_{,tt} - \epsilon^{-2} p_{,jj} = [(1+\rho) u_i u_j]_{,ij}, \tag{16}$$

where $\epsilon^2 = \gamma M_t^2$ and $M_t = \tilde{u}/c_\infty$ where $\tilde{u} = 2k/3 = \langle u_j u_j \rangle/3$ and $c_\infty^2 = \gamma P_\infty/\rho_\infty$. Note that the choice of time scales is determined by the energy-containing scales of the motion. A meaningful balance, giving bounded derivatives on the velocity, is established if $p \sim \epsilon^2$. It then follows that $\rho \sim \epsilon^2$ also. The conventional definition of the Mach number, in accordance with the acoustics literature from which some of our ideas are drawn, is used. It is the small parameter that emerges naturally in the relevant non-dimensionalization of the compressible equations. This means that $M_t = (2k/3)^{1/2}/c$ is a factor 0.577 smaller than the Mach numbers defined using $q^2 = \langle u_j u_j \rangle$, and $M_t = q/c$. Expansions of the form

$$p = \epsilon^2\,[p_1 + \epsilon^2 p_2 + \ldots, \tag{17}$$

$$\rho = \epsilon^2\,[\rho_1 + \epsilon^2 \rho_2 + \ldots, \tag{18}$$

$$u_i = v_i + \epsilon^2\,[w_i + \epsilon^2 w_{2i} + \ldots \tag{19}$$

are chosen. Inserting the expansions into the equations produces, to the lowest or zeroth order, the incompressible equations

$$v_{i,t} + v_p v_{i,p} + p_{1,i} = 0, \tag{20}$$

$$p_{1,jj} = -(v_i v_j)_{,ij}, \tag{21}$$

$$\gamma\rho_1 = p_1, \tag{22}$$

where $v_{i,i} = 0$. The correction for the compressibility of the flow does not involve a wave equation on the inner spatial and temporal scales of the turbulence. The next order equations are

$$\rho_{1,t} + v_p \rho_{1,p} = - w_{k,k}\,, \tag{23}$$
$$w_{i,t} + v_p w_{i,p} + w_p v_{i,p} + p_{2,i} = \rho_1 (v_{i,t} + v_p v_{i,p}), \tag{24}$$
$$p_2 - \gamma \rho_2 = \tfrac{1}{2}\gamma(\gamma - 1)\rho_1^2, \tag{25}$$
$$- p_{2,jj} = (w_i v_j + w_j v_i + \rho_1 v_i v_j)_{,ij} - \rho_{1,tt}\,. \tag{26}$$

In the near-field region, which is small with respect to an acoustic length scale, the compressible pressure is felt, effectively instantaneously. Additional amplification of some of these ideas can be found in the insightful paper by Crow (1970). This completes the derivation of the evolution equations for the inner expansion. For the single-point turbulence closures for compact flows, the outer solution, associated with the sound propagation problem, is not required.

The zeroth-order equations show that the density fluctuations are given by the pressure fluctuations, $\gamma \rho_1 = p_1$. The first-order inner expansion of the continuity equation now becomes a *diagnostic* relation for the fluctuating dilatation,

$$- \gamma d = p_{,t} + v_k p_{,k}\,. \tag{27}$$

The subscript on p_1 has been dropped. Evidently one does not need a solution to the evolution equation for the compressible velocity field, w_i, to obtain its dilatation. This is a very nice result† that forms the basis of the present pseudo-sound analysis. The dilatation is diagnostically related to the *local* fluctuations of the pressure and velocity; it is the rate of change of the incompressible pressure field following a fluid particle. Constitutive relations for the pressure–dilatation and the variance of the dilatational can be found by taking the appropriate moment of the fluctuating dilatation equation to produce, dropping the subscript,

$$-2\gamma \langle pd \rangle = \langle pp \rangle_{,t} + \langle v_k pp \rangle_{,k}\,, \tag{28}$$
$$\gamma^2 \langle dd \rangle = \langle \dot{p}\dot{p} \rangle + 2\langle \dot{p} v_k p_{,k} \rangle + \langle v_k p_{,k}\, v_q p_{,q} \rangle. \tag{29}$$

For a homogeneous or quasi-normal field the flux term $\langle v_k pp \rangle_{,k} = 0$. The compressibility effects, as manifested in the dilatational covariances, have now been directly linked, to leading order, to the solenoidal parts of the velocity field. Both Zeman (1991*a, b*) and Sarkar *et al.* (1991) have derived similar looking equations for the pressure dilatation. Their equations follow from a squaring of the Reynolds decomposition of the continuity equation and ensemble averaging. Their results linked the pressure–dilatation to the variance of the full pressure field. The present relationship is substantially different: the perturbation analysis has shown that the major contribution to the pressure variance field, at low M_t, is the pressure related to the solenoidal portions of the velocity field. This pressure has been called the pseudo-pressure (Ribner 1962; Ffowcs Williams 1969) and satisfies a Poisson equation for which traditional methods of classical linear mathematics are available. This fact is now exploited to obtain expressions for the dilatational covariances.

† A similar expression in different contexts with different assumptions has been obtained independently by both Girimaji (1995, personal communication) and Crow (1970).

3. Analysis

Analytical expressions for the isotropic portions of the pressure dilatation, and the variance of the dilatation are now derived.

3.1. *The pressure–dilatation in homogeneous isotropic turbulence*

Expressions for the dilatational covariances for isotropic turbulence without mean deformation are first obtained. This is analogous to the slow pressure component of turbulence models for pressure–strain covariances. Batchelor (1951) has obtained a representation for the pressure variance, $\langle pp \rangle$, in isotropic incompressible turbulence. A simpler Green's function method, following Kraichnan (1956), is used here. The non-dimensional pressure satisfies the Poisson equation: $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$. The two-point pressure variance obeys

$$\langle p(x,t)p(x',t)\rangle_{,jjp'p'} = \langle v_i v_j v'_p v'_q \rangle_{,ijp'q'} \tag{30}$$

which can be written in terms of the separation variable $r_i = x'_i - x_i$. Following the usual methods for translationally invariant random processes,

$$\langle p(x,t)p(x',t)\rangle_{,jjpp} = \langle v_i v_j v'_p v'_q \rangle_{,ijpq}\,. \tag{31}$$

The Green's function for the equation is $-(1/8\pi) \mid r' - r \mid$ and the solution is expressed as

$$\langle pp' \rangle = -\frac{1}{8\pi} \int \langle v_i v_j v'_p v'_q \rangle_{,ijpq} \mid r' - r \mid \mathrm{d}^3 r'. \tag{32}$$

Now $\langle p(x,t)p(x',t)\rangle = \langle pp'\rangle(r)$. The quasi-normal assumption is used to relate the fourth-order moments of the velocity field to its second-order moments. A discussion of the adequacy of the quasi-normal assumption is relegated to §5 where all the assumptions made in this analysis are addressed. The assumption allows

$$\langle v_i v_j v'_p v'_q \rangle = \langle v_i v_j \rangle \langle v'_p v'_q \rangle + \langle v_i v'_p \rangle \langle v_j v'_q \rangle + \langle v_i v'_q \rangle \langle v'_p v_j \rangle. \tag{33}$$

The definition for the correlation, $\langle v_i v'_j \rangle = \frac{2}{3} k R_{ij}(r)$, where $k = \frac{1}{2}\langle v_j v_j \rangle$, is used to obtain

$$\langle v_i v_j v'_p v'_q \rangle_{,ijpq} = 2\ \langle v_i v'_p \rangle_{,jq}\ \langle v_j v'_q \rangle_{,ip} = 2\ (\tfrac{2}{3}k)^2 R_{ip,jq} R_{jq,ip}. \tag{34}$$

Upon application of continuity, $R_{ij,j}(r) = 0$, the pressure variance becomes

$$\langle pp' \rangle = -2\ (\tfrac{2}{3}k)^2 \frac{1}{8\pi} \int R_{ip,jq} R_{jq,ip} \mid r' - r \mid \mathrm{d}^3 r'. \tag{35}$$

The expression is exact for a general R_{ij}. It is virtually impossible to do these integrations for a general R_{ij}. The leading-order isotropic form of R_{ij} is taken. This procedure, as it is applied on subsequent occasions, is further discussed in §5. The integral can then be written in terms of the longitudinal correlation function. Following von Kármán & Howarth (1938), the longitudinal correlation, $\langle v_1(0) v_1(r) \rangle = \langle v_1 v_1 \rangle f(r) = \frac{2}{3} k f(r) = \frac{2}{3} k R_{11}$, allows the isotropic portion of the two-point correlation to be written as $R_{ij} = -(r_i r_j / 2r) f' + (f + \frac{1}{2} r f')\delta_{ij}$. Following Batchelor's (1951) development, the fourth-order two-point correlation can be expressed as

$$\begin{aligned} R_{ip,jq} R_{jq,ip} &= 2 \left[2f''^2 + 2f'f''' + \frac{10}{r} f'f'' + \frac{3}{r^2} f'^2 \right] \\ &= 2\, \frac{1}{\xi^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \left[\frac{1}{\xi} \frac{\mathrm{d}}{\mathrm{d}\xi} (\xi^3 f'^2) \right]. \end{aligned} \tag{36}$$

The integrand can be written in terms of the scalar function $f(\xi)$, where $\xi = r/\ell$ is the non-dimensional spatial coordinate such that $\int f(\xi)\mathrm{d}\xi = 1$. Inserting into the integrand and applying integration by parts successively produces, returning to dimensional variables, the Batchelor (1951) result:

$$\langle pp\rangle = 2\,(\tfrac{2}{3}k)^2\rho_\infty^2 \int_0^\infty \xi f'^2(\xi)\mathrm{d}\xi = \tfrac{8}{9}\rho_\infty^2 k^2 I_1^s, \tag{37}$$

where $I_1^s = \int_0^\infty \xi f'^2(\xi)\mathrm{d}\xi$. Inserting the result into the constitutive relation for the pressure–dilatation, $-2\gamma\langle pd\rangle = (\mathrm{D}/\mathrm{D}t)\langle pp\rangle$, produces, in dimensional quantities, the following expression for the slow pressure–dilatation:

$$\langle pd\rangle^s = -\tfrac{2}{3}I_1^s\,(\mathrm{D}/\mathrm{D}t)\,[\bar{\rho}M_t^2 k]. \tag{38}$$

Here $\mathrm{D}/\mathrm{D}t$ is the substantial derivative following a mean fluid particle. The reference density and pressure have been replaced by the local mean density and pressure.

3.2. *The dilatational variance in isotropic turbulence*

The quasi-normal form of the constitutive relation for the variance of the dilatation is

$$\gamma^2\langle dd\rangle = \langle \dot{p}\dot{p}\rangle + \langle v_p p_{,p}\, v_q p_{,q}\rangle. \tag{39}$$

Starting, once again, from the non-dimensional Poisson equation for the leading-order pressure field, $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$, an equation similar to the two-point variance of the pressure derived above can be obtained for the variance of the time derivative of the pressure: thus

$$\langle \dot{p}(x,t)\dot{p}(x',t)\rangle_{,jjpp} = \langle (v_i v_j)_{,t}\,(v'_p v'_q)_{,t}\rangle_{,ijpq} = 4\langle \dot{v}_i v_j \dot{v}'_q v'_p\rangle_{,ijpq} \tag{40}$$

after expanding the products of the time derivatives. The equation for the variance becomes

$$\langle \dot{p}(x,t)\dot{p}(x',t)\rangle_{,jjpp} = 4\,\langle \dot{v}_i\dot{v}'_p\rangle_{,jq}\,\langle v_j v'_q\rangle_{,ip} \tag{41}$$

using the quasi-normal assumption. The fact that $\langle \dot{v}_j v'_q\rangle = 0$ for homogeneous isotropic turbulence, as can be seen from the Navier–Stokes equations and application of continuity, has been used. The tensor $\langle \dot{v}_i\dot{v}'_j\rangle$ can be written in terms of the correlation function, $\langle \dot{v}_i\dot{v}'_j\rangle = \langle \dot{v}\dot{v}\rangle R_{1ij}$ which can be rewritten in terms of the scalar correlation function, f_1, following the usual procedures as $R_{1ij} = -(r_i r_j/2r)f'_1 + (f_1 + \tfrac{1}{2}rf'_1)\delta_{ij}$ to produce $\langle \dot{v}_i\dot{v}'_i\rangle = \langle \dot{v}\dot{v}\rangle[3f_1 + rf'_1] = \langle \dot{v}\dot{v}\rangle r^{-2}(r^3 f_1)'$. For an isotropic tensor there is only one scalar function, say $\langle \dot{v}\dot{v}\rangle$. This will be taken to be $\langle \dot{v}\dot{v}\rangle = \langle \dot{v}_1\dot{v}_1\rangle$. The bi-harmonic equation for the variance of the time derivative of the pressure becomes

$$\langle \dot{p}\dot{p}'\rangle_{,jjpp} = 8\,\tfrac{2}{3}k\langle \dot{v}\dot{v}\rangle\frac{1}{r^4}\left[\frac{1}{r^3}(rf'f'_1)'\right]'. \tag{42}$$

Using the Green's function method and integrating by parts produces, in dimensional form,

$$\langle \dot{p}\dot{p}\rangle = 4\,\rho_\infty^2\,\tfrac{2}{3}k\langle \dot{v}\dot{v}\rangle \int_0^\infty \xi f' f'_1\,\mathrm{d}\xi. \tag{43}$$

An expression for the two-point variance of the acceleration, $\langle \dot{v}\dot{v}\rangle f'_1$ is required. Little is known about the longitudinal correlation of the acceleration. The Navier–Stokes equations can be used to obtain an equation relating the acceleration correlation,

f_1, to f, the longitudinal correlation of the velocity correlation.† The dynamical equations of the inviscid portions of the motion, in the absence of a mean velocity field, can be used to produce the following equation for the two-point covariance of the acceleration:

$$\langle \dot{v}_i \dot{v}_i' \rangle = -\rho_\infty^{-2} \langle pp' \rangle_{,jj} - \langle v_i v_j v_i' v_k' \rangle_{,jk} . \tag{44}$$

The quasi-normal assumption for the last term on the right in the evolution equation for the two-point acceleration produces

$$\begin{aligned} \langle v_i v_j v_i' v_k' \rangle_{,jk} &= [\langle v_i v_j \rangle \langle v_i' v_k' \rangle + \langle v_i v_i' \rangle \langle v_j v_k' \rangle + \langle v_i v_k' \rangle \langle v_i' v_j \rangle]_{,jk} \\ &= (\tfrac{2}{3}k)^2 \frac{1}{r^2} [r^3 (ff'' - \tfrac{1}{2}f'^2 + (4/r)ff')]' \end{aligned} \tag{45}$$

after substituting in terms of the longitudinal correlation. For operations on functions of r, the Laplacian can be written $\langle pp' \rangle_{,jj} = r^{-2}(\mathrm{d}/\mathrm{d}r)(r^2(\mathrm{d}/\mathrm{d}r)\langle pp' \rangle)$ and the dynamical equation for the two-point acceleration becomes, after one integration,

$$\langle \dot{v}\dot{v} \rangle f_1 = -\frac{1}{\rho_\infty^2} \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \langle pp' \rangle - (\tfrac{2}{3}k)^2 (ff'' - \tfrac{1}{2}f'^2 + (4/r)ff'). \tag{46}$$

From the expression for the two-point covariances for the pressure (Batchelor 1951), the following expression can be derived:

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \langle pp' \rangle = -4\rho_\infty^2 \left(\tfrac{2}{3}k\right)^2 \int_r^\infty \frac{1}{r'} f'^2 \mathrm{d}r'. \tag{47}$$

Inserting Batchelor's expression in the equation for f_1 and taking the derivative produces the quantity required,

$$\langle \dot{v}\dot{v} \rangle f_1' = -\left(\tfrac{2}{3}k\right)^2 \left(ff''' + \frac{4}{r} ff'' + \frac{8}{r} f'f' - \frac{4}{r^2} ff' \right). \tag{48}$$

Inserting the expression for $\langle \dot{v}\dot{v} \rangle f_1'$ into the variance, $\langle \dot{p}\dot{p} \rangle = 4\rho_\infty^2 \frac{2}{3} k \langle \dot{v}\dot{v} \rangle \int_0^\infty \xi f'(\xi) f_1'(\xi) \mathrm{d}\xi$ produces

$$\langle \dot{p}\dot{p} \rangle = 4\, \rho_\infty^2 \left(\tfrac{2}{3}k\right)^3 \frac{1}{\ell^2} I_2^s = \frac{9}{\alpha^2} \rho_\infty^2 \left(\tfrac{2}{3}k\right)^2 \frac{\varepsilon^2}{k^2} I_2^s = \frac{4}{\alpha^2} \rho_\infty^2 \varepsilon^2 I_2^s, \tag{49}$$

where $I_2^s = \int_0^\infty \xi f' [ff''' + (4/\xi) ff'' + (8/\xi) f'f' - (4/\xi^2) ff'] \mathrm{d}\xi$. The characteristic velocity fluctuation will be taken to be $\tilde{u}^2 = \frac{2}{3}k$ and the Kolmogorov scaling $\varepsilon = \alpha(\frac{2}{3}k)^{3/2}/\ell$ has been used. This is the second phenomenological assumption made: the first is the quasi-normal assumption. Both of these assumptions are discussed further in §5.

The fourth-order moments in the constitutive expression $\gamma^2 \langle dd \rangle = \langle \dot{p}\dot{p} \rangle + \langle v_p p_{,p}\, v_q p_{,q} \rangle$ are now treated. Beginning with the two-point statistic and writing it as a function of the separation distance, r_i,

$$\begin{aligned} \langle v_p p_{,p}\, v_q' p'_{,q'} \rangle &= -\langle v_p p v_q' p' \rangle_{,pq} \\ &= -[\langle v_k v_q' \rangle \langle pp' \rangle + \langle v_p p \rangle \langle v_q' p' \rangle + \langle v_p p' \rangle \langle v_q' p \rangle]_{,pq} \\ &= -\langle v_k v_q' \rangle \langle pp' \rangle_{,pq} \end{aligned} \tag{50}$$

where continuity, $\langle v_p v_q' \rangle_{,p} = 0$, and the fact that an isotropic vector is zero have been used. Further manipulations and setting $r = 0$ produces

$$-\langle v_k v_q \rangle \langle pp \rangle_{,pq} = -\tfrac{2}{3}k \langle pp \rangle_{,pp} - 2kb_{pq} \langle pp \rangle_{,pq} \tag{51}$$

† This development was indicated by Y. Zhou at ICASE.

where b_{pq} is the anisotropy tensor, $b_{ij} = \langle v_i v_j \rangle / 2k - \frac{1}{3}\delta_{ij}$. A theory including the contribution of the anisotropy is possible but requires the pressure variance Hessian. Expressing the two-point covariance in terms of its longitudinal correlation function and performing the appropriate differentiations of $\langle pp' \rangle = \langle pp \rangle P(r)$ produces

$$-\tfrac{2}{3}k\langle pp\rangle_{,pp} = -\tfrac{2}{3}k\langle pp\rangle 3P_0''. \tag{52}$$

The second derivative of Batchelor's solution for the two-point pressure variance can be used to show that $P_0'' = -(4/\ell^2)I_3^s$ where $I_3^s = \int_0^\infty (1/\xi) f'^2 \mathrm{d}\xi$. The fourth-order moment can be written as

$$\langle v_p p_{,p}\, v_q p_{,q} \rangle = \tfrac{2}{3}k\langle pp\rangle \frac{12}{\ell^2}\, I_3^s = 16\rho_\infty^2(\tfrac{2}{3}k)^2\, \frac{k}{\ell^2} I_1^s I_3^s = \frac{54}{\alpha^2}\rho_\infty^2(\tfrac{2}{3}k)^2\, \frac{\varepsilon^2}{k^2} I_1^s I_3^s. \tag{53}$$

The earlier result for the pressure variance and the Kolmogorov scaling $\varepsilon = \alpha(\frac{2}{3}k)^{3/2}/\ell$ have been used. These results are substituted into the constitutive equation for the variance of the dilatation – $\gamma^2\langle dd\rangle = \langle \dot{p}\dot{p}\rangle + \langle v_p p_{,p}\, v_q p_{,q}\rangle$ – to obtain the following simple expression for the slow portion of the variance of the dilatation:

$$\langle dd\rangle^s = \frac{9}{\alpha^2} M_t^4 \left(\frac{\varepsilon}{k}\right)^2\, [I_2^s + 6 I_1^s I_3^s]. \tag{54}$$

The integrals are, typically, order-one quantities. Y. Zhou (1994, personal communication) has determined their value from high Reynolds number wind tunnel data. These values are given in the Appendix. The compressible dissipation is defined as $\bar{\rho}\varepsilon_c = \frac{4}{3}\langle\mu\rangle\langle dd\rangle$ and the model can be put in a form relevant to the kinetic energy equation:

$$\varepsilon_c^s = \frac{16}{3\alpha^2}\frac{M_t^4}{R_t}\, \varepsilon_s\, [I_2^s + 6 I_1^s I_3^s]. \tag{55}$$

The traditional and physically meaningful definition of the turbulent Reynolds number (Tennekes & Lumley 1972), based on a characteristic velocity, $\tilde{u}^2 = \frac{2}{3}k$, and the Kolmogorov outer length scale, ℓ, is used: $R_t = \tilde{u}\ell/\langle v\rangle = 4k^2/9\langle v\rangle\varepsilon$. This is a factor 9 smaller than the more recent definition adopted by the DNS community.

3.3. *The pressure–dilatation due to mean velocity gradients*

An expression for the pressure–dilatation in the presence of a general homogeneous mean velocity gradient and no mean bulk dilatation is now derived. The constitutive relation for the pressure–dilatation is

$$-2\gamma\langle pd\rangle = \frac{\mathrm{D}}{\mathrm{D}t}\langle pp\rangle. \tag{56}$$

The velocity field is partitioned according to the Reynolds decomposition $V_i + v_i$; the upper case denotes a steady mean velocity field with constant gradients, the lower case will continue to indicate the fluctuating field. The mean strain and rotation tensors are $S_{ij} = \frac{1}{2}\,[V_{i,j} + V_{j,i}]$, $W_{ij} = \frac{1}{2}\,[V_{i,j} - V_{j,i}]$; W^2 and S^2 denote the traces of the squares of these matrices. The non-dimensional Poisson equation, $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$, now involves the mean velocity gradient:

$$p(x,t)_{,jj} = -(v_i V_j + V_i v_j + v_i v_j)_{,ij}\,.$$

Multiplying this equation by a similar Poisson equation for $p(x',t)$ and averaging produces

$$\langle p(x,t)p(x',t)\rangle_{,j'j'qq} = 4V_{i,j}\, V_{p,q'}\, \langle v_{j,i}\, v'_{q,p'}\rangle + \langle v_i v_j v'_p v'_q\rangle_{,ijp'q'}\,.$$

The last term, which represents the slow pressure contribution, was obtained in a previous section. Expressing the differential equation in terms of the spatial separation, r_i, produces a biharmonic equation for the two-point pressure variance

$$\langle pp' \rangle_{,jjqq} = 4V_{i,j}\, V_{p,q}\, \langle v_j v'_q \rangle_{,ip}\,. \tag{57}$$

The Green's function method produces the following solution:

$$\langle pp' \rangle(r) = 4V_{i,j}\, V_{p,q}\, \frac{1}{8\pi} \int \langle v_j v'_q \rangle_{,ip} \, |r - r'| \mathrm{d}^3 r' = 4V_{i,j}\, V_{p,q}\, I_{jqip}(r). \tag{58}$$

The expression for the pressure variance, at this point, involves no assumptions about the two-point correlation function, $\langle v_j v'_q \rangle = \frac{2}{3}kR_{jq}$. The pressure variance is known once a representation for the integral I_{jqip} is found. For a class of turbulent flows a tensor polynomial in the anisotropy tensor is a suitable approximation for I_{jqip}. Lumley (1970), Ristorcelli, Lumley & Abid (1995), Ristorcelli (1996) discuss issues related to this assumption. Here, only the leading-order term in such a polynomial will be retained for the purpose of understanding the physics and obtaining scalings for the scalar effects of compressibility. Higher-order terms introduce an element of empiricism while not changing, conceptually, any of the results. This is discussed in §5. A fourth-order isotropic tensor with the proper symmetry and satisfying continuity, $I_{jiip} = 0$ is

$$I_{jqip} = A_1^r\ [\delta_{jq}\delta_{ip} - \tfrac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})]$$

where

$$A_1^r = \frac{2}{15} I_{jjii} = \frac{2}{15}\frac{1}{8\pi} \int \langle v_j v'_j \rangle_{,ii}\, r' \mathrm{d}^3 r' = \frac{1}{15}\frac{2k}{3}\ell^2 I_1^r. \tag{59}$$

Expressing the integrand in terms of the longitudinal correlation in the normalized coordinate, $\langle v_j v'_j \rangle_{,ii} = \frac{2}{3}k\ \xi^{-2}[\xi^3 f''' + 7\xi^2 f'' + 8\xi f']$. The facts that $\langle v_j v'_j \rangle = \frac{2}{3}k[rf' + 3f] = r^{-2}(\mathrm{d}/\mathrm{d}r)(r^3 f)$ and, that in spherical coordinates, the Laplacian is $\nabla^2 = r^{-2}(\mathrm{d}/\mathrm{d}r)r^2(\mathrm{d}/\mathrm{d}r)$ have been used. It is also possible to integrate by parts, allowing the integrand to be expressed in lower-order derivatives for more accurate computation from experimental data. Thus

$$\begin{aligned} A_1^r &= \frac{1}{15}\frac{2}{3}k\ell^2 \int_0^\infty \xi^2[\xi^2 f''' + 7\xi f'' + 8f']\mathrm{d}\xi \\ &= \frac{1}{15}\frac{2}{3}k\ell^2 \int_0^\infty \xi\ \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^{-2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi^3 f)\right)\right)\mathrm{d}\xi \\ &= \frac{2}{15}\frac{2}{3}k\ell^2 \int_0^\infty \xi f \mathrm{d}\xi = \frac{1}{15}\frac{2}{3}k\ell^2 I_1^r. \end{aligned} \tag{60}$$

Note that the coefficient A_1^r has been related to a quantity related to the turbulence – it is not an empirical coefficient. The leading-order solution for the rapid pressure variance in an arbitrary three-dimensional mean velocity gradient is then expressed as

$$\langle pp \rangle^r = \tfrac{1}{15}\rho_\infty^2 \tfrac{2}{3}k\ell^2\ [3S^2 + 5W^2]\ I_1^r. \tag{61}$$

The integral has dimensions of a characteristic correlation area: the rapid pressure contribution to the pressure variance will vary according to the spatial scale of the turbulence, unlike the slow pressure contribution given above. This dependence on the spatial scale was noted by Kraichnan (1956), who produced a leading-order expression for pressure fluctuations in a unidirectional shear. The results here extend Kraichnan's

(1956) results for a simple shear to an arbitrary (three-dimensional) quasi-stationary mean deformation.

The results are substituted into the constitutive equation and the rapid portion of the pressure–dilatation covariance in dimensional variables becomes

$$\langle pd\rangle^r = -\frac{1}{30}\, I_1^r \frac{\mathrm{D}}{\mathrm{D}t}\left[\rho_\infty \tfrac{2}{3}k \frac{\ell^2}{c_\infty^2}\; 3S^2[1+\tfrac{5}{3}R^2]\right], \tag{62}$$

where $R^2 = W^2/S^2$. Note the appearance of the quantity $S\ell/c_\infty$; the dependence of the compressibility effects on a deformation-rate Mach number have been indicated by Lele (1994) and Sarkar (1995). The expression is recast in terms of the turbulent Mach number and the rapid component of the pressure–dilatation covariance becomes

$$\langle pd\rangle^r = -\frac{1}{30}\, I_1^r\, \frac{\mathrm{D}}{\mathrm{D}t}\, [\bar{\rho} M_t^2 \ell^2\; 3S^2[1+\tfrac{5}{3}R^2]]. \tag{63}$$

Two effects contribute to the pressure–dilatation covariance: one due to the rate of change of kinetic energy ($M_t^2 \sim k$) and the other due to changes in the area of the correlation. Unlike the slow pressure–dilatation however, the rapid pressure–dilatation does not always have the opposite sign to the growth of kinetic energy but now depends on the rate of increase of the area of the correlation, ℓ^2. The well-established Kolmogorov scaling $\varepsilon = \alpha(\frac{2}{3}k)^{3/2}/\ell$ (see §5), where α is a flow-dependent quantity, is used to close the expression:

$$\langle pd\rangle^r = -\tfrac{1}{30}\,(\tfrac{2}{3})^3 I_1^r\, \alpha^2 \frac{\mathrm{D}}{\mathrm{D}t}\, [\bar{\rho} k M_t^2 \hat{S}^2 [3+5R^2]]. \tag{64}$$

The quantities with a carat are the relative strain and rotation rates, e.g. $\hat{S}^2 = (Sk/\varepsilon)^2$.

3.4. *The dilatational variances due to mean velocity gradients*

In the constitutive relationship for the variance of the dilatation the time derivative is replaced by the mean advective derivative, $\mathrm{D}/\mathrm{D}t = (\;)_{,t} + V_k(\;)_{,k}$. Carrying the substantial derivative as part of the time derivative term involves no approximation and follows quite naturally from the Reynolds decomposition. The development preserves Galilean invariance. The quasi-normal form of the constitutive relation for the pressure–dilatation is

$$\gamma^2\langle dd\rangle = \langle \mathring{p}\mathring{p}\rangle + \langle v_p p_{,p}\, v_q p_{,q}\rangle. \tag{65}$$

The small circle indicates the mean convective derivative; thus $\langle \mathring{p}\mathring{p}\rangle = \langle(\mathrm{D}p/\mathrm{D}t)(\mathrm{D}p/\mathrm{D}t)\rangle$.

For the convenience of the presentation the two contributions to the variance of the dilatation will be denoted $\gamma^2\langle dd\rangle_1 = \langle \mathring{p}\mathring{p}\rangle$ and $\gamma^2\langle dd\rangle_2 = \langle v_p p_{,p}\, v_q p_{,q}\rangle$. Applying the Reynolds decomposition to the non-dimensional form of the Poisson equation for pressure, $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$, and taking the appropriate derivatives and dropping the terms quadratic in the fluctuating velocities (treated in an earlier section) produces

$$\mathring{p}\,(x,t)_{,jj} = -(\mathring{v}_i\, V_j + V_i\, \mathring{v}_j)_{,ij} = -2V_{i,j}\, \mathring{v}_{j,i}\,. \tag{66}$$

Multiplying this by the Poisson equation for $\mathring{p}\,(x',t)$ and averaging produces the biharmonic equation for the two-point pressure covariance

$$\langle \mathring{p}\,(x,t)\, \mathring{p}\,(x',t)\rangle_{,j'j'qq} = 4V_{i,j}\, V_{p,q'}\, \langle \mathring{v}_{j,i}\, \mathring{v}'_{q,p'}\rangle = 4V_{i,j}\, V_{p,q}\, \langle \mathring{v}_j \mathring{v}'_q\rangle_{,ip} \tag{67}$$

in terms of the separation variable, r_i. The Green's function solution procedure produces the following representation for the two-point covariance:

$$\langle \overset{\circ}{p}\overset{\circ}{p}' \rangle = 4V_{i,j}\, V_{p,q}\, \frac{1}{8\pi} \int \langle \overset{\circ}{v}_j \overset{\circ}{v}'_q \rangle_{,ip}\, |r - r'| \mathrm{d}^3 r' = 4V_{i,j}\, V_{p,q}\, I_{jqip}(r). \tag{68}$$

It is the variance (at $r = 0$) that is required. The fourth-order tensor, neglecting higher-order corrections for anisotropy, is represented as the isotropic tensor

$$\left.\begin{aligned} I_{jqip} &= A_2^r[\delta_{jq}\delta_{ip} - \tfrac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})], \\ A_2^r &= \frac{\ell^2}{15}\int_0^\infty \langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,pp}\, \xi^3 \mathrm{d}\xi = \frac{\ell^2}{15}\frac{2k}{3} I_2^r. \end{aligned}\right\} \tag{69}$$

If the acceleration correlation were known it would be a simple matter to show that the integrand is given by $\langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,ii} = \langle \overset{\circ}{v}\overset{\circ}{v} \rangle [\xi^3 f_1''' + 7\xi^2 f_1'' + 8\xi f_1']\xi^{-2}$. (For isotropic functions $\langle \dot{v}\dot{v} \rangle = \frac{1}{3}\langle v_j v_j \rangle$ is arbitrary; it can be understood as $\langle \dot{v}\dot{v} \rangle = \langle \dot{v}_1 \dot{v}_1 \rangle$.) This is not the case and an expression for f_1 in terms of f is required. The Navier–Stokes equations are used to obtain an expression for the integral, $\int_0^\infty \langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,pp}\, \xi^3 \mathrm{d}\xi$. Taking the equation for $-\overset{\circ}{v}_i = v_k V_{i,k} + v_k v_{i,k} + p_{,i}$, and multiplying it by a similar equation for $\overset{\circ}{v}'_j$, averaging and taking the trace produces, in the r_i coordinate, gives

$$\begin{aligned} \langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle = &-V_{i,k}\, V_{i,q}\, \langle v_k v'_q \rangle - \langle p p' \rangle_{,jj} - [V_{i,k}\, \langle v_k v'_q v'_i \rangle_{,q} + V_{i,q}\, \langle v_k v_i v'_q \rangle_{,k}] \\ &- [\langle p v'_k v'_i \rangle_{,ik} + \langle p' v_i v_k \rangle_{,ik}] - [V_{i,k}\, \langle v_k v'_q v'_i \rangle_{,q} + V_{i,q}\, \langle v_k v_i v'_q \rangle_{,k}] \\ &- \langle v_j v_k v'_q v'_j \rangle_{,kq}\, . \end{aligned}$$

The two-point triple covariances are zero for homogeneous isotropic turbulence and the fourth-order correlation was treated in a previous section. The equation yields, after taking the Laplacian, the quantity sought:

$$\langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,pp} = -V_{i,k}\, V_{i,q}\, \langle v_k v'_q \rangle_{,pp} - \langle p p' \rangle_{,jjpp}\, . \tag{70}$$

In a previous section it was shown that the two-point pressure variance satisfied the biharmonic equation: $\langle p p' \rangle_{,jjqq} = 4V_{i,j}\, V_{p,q}\, \langle v_j v'_q \rangle_{,ip}$. Thus

$$\langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,pp} = -V_{i,k}\, V_{i,q}\, \langle v_k v'_q \rangle_{,pp} - 4V_{i,j}\, V_{p,q}\, \langle v_j v'_q \rangle_{,ip}\, , \tag{71}$$

which upon multiplication by ξ^3 and integration produces the desired result for $\int_0^\infty \langle \overset{\circ}{v}_j \overset{\circ}{v}'_j \rangle_{,pp}\, \xi^3 \mathrm{d}\xi$ in the definition of I_2^r:

$$\tfrac{2}{3} k I_2^r = -V_{i,k}\, V_{i,q}\, I_{kq} - 4V_{i,j}\, V_{p,q}\, I_{jqip} \tag{72}$$

and I_2^r is seen to be related to the two integrals, $I_{kq} = \int_0^\infty \langle v_k v'_q \rangle_{,pp}\, \xi^3 \mathrm{d}\xi$, and $I_{jqip} = \int_0^\infty \langle v_j v'_q \rangle_{,ip}\, \xi^3 \mathrm{d}\xi$. The isotropic portions of these tensors are related to an earlier integral, I_1^r, defined in the previous section. The tensors have the following representations:

$$\begin{aligned} I_{jqip} &= \tfrac{2}{15} I_1^r\, [\delta_{jq}\delta_{ip} - \tfrac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})], \\ I_{jq} &= \tfrac{1}{3} I_1^r\, \delta_{jq}. \end{aligned}$$

Inserting these expressions into the equation for I_2^r produces an expression for the two-point acceleration correlation integral in terms of the two-point velocity correlation integrals: $I_2^r = \frac{1}{30}\, I_1^r\, [13S^2 + 15W^2\,]$, and the rapid-pressure variance becomes

$$\langle \mathring{p}\mathring{p}\rangle^r = \tfrac{1}{15}\tfrac{1}{30}\,[3S^2 + 5W^2]\,[13S^2 + 15W^2\,]\,I_1^r \ell^2\,\tfrac{2}{3}k. \tag{73}$$

Substituting $\ell = \alpha(2k/3)^{3/2}/\varepsilon$, and inserting into $\gamma^2\langle dd\rangle_1 = \langle \mathring{p}\mathring{p}\rangle$ which is related to the local dilatational dissipation by $\varepsilon_{c1}^r = \frac{4}{3}\nu\langle dd\rangle_1 = \langle \mathring{p}\mathring{p}\rangle$ produces

$$\varepsilon_{c1}^r = \left(\tfrac{1}{15}\right)^2\left(\tfrac{2}{3}\right)^5\,\frac{M_t^4}{R_t}\,\varepsilon_s\,[3\hat{S}^2 + 5\hat{W}^2]\,[13\hat{S}^2 + 15\hat{W}^2\,]\,\alpha^2 I_1^r \tag{74}$$

after accounting for the non-dimensionalizations employed.

An expression for the fourth-order moment, $\langle v_p p_{,p}\, v_q p_{,q}\,\rangle$, appearing in $\langle dd\rangle_2$ is now sought. In a previous section it was seen that under the quasi-normal and isotropic approximations $\langle v_p p_{,p}\, v_q p_{,q}\,\rangle = -\frac{2}{3}k\langle pp\rangle_{,qq}$. The Green's function method produces

$$\langle pp'\rangle_{,jj} = 4V_{i,j}\,V_{p,q}\,\frac{-1}{4\pi}\int \langle v_j v_q'\rangle_{,ip}\,\frac{\mathrm{d}^3 r'}{|r-r'|} = 4V_{i,j}\,V_{p,q}\,I_{jqip}(r). \tag{75}$$

The biharmonic equation for the pressure variance from the previous section has been used. Following the usual procedures with $I_{jqip} = A_3^r\,[\delta_{jq}\delta_{ip} - \frac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})]$ produces

$$\langle pp\rangle_{,jj} = \tfrac{2}{15}\tfrac{2}{3}k\,[3S^2 + 5W^2]I_3^r, \tag{76}$$

where

$$A_3^r = \tfrac{2}{15}I_{jjii} = \tfrac{2}{15}\tfrac{2}{3}kI_3^r. \tag{77}$$

Using the facts that $\langle v_j v_j'\rangle = \frac{2}{3}k[rf' + 3f] = r^{-2}(\mathrm{d}/\mathrm{d}r)(r^3 f)$ and $\nabla^2 = r^{-2}(\mathrm{d}/\mathrm{d}r)r^2(\mathrm{d}/\mathrm{d}r)$ produces $I_3^r = -\int_0^\infty \xi^2 f''' + 7\xi f'' + 8f'\,\mathrm{d}\xi$. Reflection on the integrand will show that it is suitable for application of Gauss's theorem: the exact result is $I_{jjii} = \langle v_j v_j\rangle = \frac{2}{3}k3f(0)$ and $I_3^r = 3$. The fourth-order moment becomes

$$\langle v_p p_{,p}\, v_q p_{,q}\,\rangle = -\tfrac{2}{3}k\langle pp\rangle_{,qq} = \tfrac{2}{15}(\tfrac{2}{3}k)^2\,[3S^2 + 5W^2]I_3^r \tag{78}$$

and thus

$$\langle dd\rangle_2 = \tfrac{2}{15}M_t^4\,[3S^2 + 5W^2]I_3^r, \tag{79}$$

which allows the second portion of the rapid dilatational dissipation to be expressed as

$$\varepsilon_{c2}^r = \tfrac{3}{5}(\tfrac{2}{3})^5\,\frac{M_t^4}{R_t}\,\varepsilon_s\,\hat{S}^2[3 + 5R^2]\,I_3^r, \tag{80}$$

where $R^2 = \hat{W}^2/\hat{S}^2$. The rapid portion of the dilatation dissipation can be written as the sum $\varepsilon_c^r = \varepsilon_{c1}^r + \varepsilon_{c2}^r$ and thus

$$\varepsilon_c^r = (\tfrac{2}{3})^5\,\frac{M_t^4}{R_t}\,\varepsilon_s\,\hat{S}^2[3 + 5R^2][\tfrac{3}{5}I_3^r + (\tfrac{1}{15})^2[13\hat{S}^2 + 15\hat{W}^2]\,\alpha^2\,I_1^r]. \tag{81}$$

This concludes the analytical development for the expressions for the local dilatational covariances.

3.5. *Summary of the results of the analysis*

As has been shown the pressure–dilatation covariance is a sum of two terms, $\langle pd\rangle = \langle pd\rangle^s + \langle pd\rangle^r$ where

$$\langle pd\rangle^s = -\tfrac{2}{3}I_1^s\,\frac{\mathrm{D}}{\mathrm{D}t}\,[\bar{\rho}M_t^2 k], \tag{82}$$

$$\langle pd\rangle^r = -\tfrac{1}{30}\,(\tfrac{2}{3})^3 I_1^r\,\alpha^2 \frac{\mathrm{D}}{\mathrm{D}t}\,[\bar\rho k M_t^2\,\hat{S}^2[3+5R^2]]. \tag{83}$$

Additional analysis is required to produce the final form of the pressure–dilatation. If the relative time scale is defined as $\mathscr{T} = [3\hat{S}^2 + 5\hat{W}^2]$, then

$$\langle pd\rangle \sim -I_{pd}\frac{\mathrm{D}}{\mathrm{D}t}\,[\bar\rho k M_t^2] - I_{pd}^r\frac{\mathrm{D}}{\mathrm{D}t}\,[\bar\rho k\ \ M_t^2\,\mathscr{T}]. \tag{84}$$

Definitions of I_{pd}^r and I_{pd} are given in the Appendix. Using the fact that $k/(c_v T) = \frac{3}{2}M_t^2\gamma(\gamma-1)$ allows the derivative of the Mach number to be eliminated in favour of derivatives of T and k:

$$\overset{\circ}{M_t^2} = M_t^2\left[\frac{\overset{\circ}{k}}{k} - \frac{\overset{\circ}{T}}{T}\right], \tag{85}$$

and the expression for the pressure–dilatation becomes

$$\langle pd\rangle \sim -I_{pd}\bar\rho M_t^2\,k\left[2\frac{\overset{\circ}{k}}{k} - \frac{\overset{\circ}{T}}{T}\right] - I_{pd}^r\bar\rho k\ \ M_t^2\,\overset{\circ}{\mathscr{T}}. \tag{86}$$

Inserting the evolution equations for T and k into the right-hand side produces the final and *almost* algebraic expression for the pressure–dilatation

$$\langle pd\rangle = -\chi_{pd}M_t^2\,[\bar\rho P_k - \bar\rho\varepsilon + T_k - \tfrac{3}{4}M_t^2\gamma(\gamma-1)(P_T + \bar\rho\varepsilon + T_T)] - \bar\rho k\ M_t^2\,\chi_{pd}^r\,\overset{\circ}{\mathscr{T}}. \tag{87}$$

The differential due to the streamwise adjustment of the relative strain, $\overset{\circ}{\mathscr{T}}$, remains. Here ε stands for the combined solenoidal and compressible contributions to the dissipation. The χ_{pd} coefficients are thus functions of the turbulent Mach number, M_t^2, and the relative strain and rotation rates, Sk/ε, Wk/ε and the Kolmogorov scaling parameter, α. The details are given in the Appendix.

The local dilatational dissipation is composed of a slow and a rapid part: $\varepsilon_c = \varepsilon_c^r + \varepsilon_c^s$ where

$$\varepsilon_c^s = \frac{16}{3\alpha^2}\frac{M_t^4}{R_t}\,\varepsilon_s\,[I_2^s + 6I_1^sI_3^s], \tag{88}$$

$$\varepsilon_c^r = (\tfrac{2}{3})^5\,\frac{M_t^4\hat{S}^2}{R_t}\,\varepsilon_s\,[3+5R^2][\tfrac{3}{5}I_3^r + (\tfrac{1}{15}\hat{S})^2[13+15R^2]\,\alpha^2\,I_1^r], \tag{89}$$

where $R^2 = W^2/S^2$ is the mean rotation to strain ratio; $R = 1$ for a pure shear. The constants I_i and α are flow-specific quantities: however, they are fully specified by their mathematical definition and can be measured in a specified flow; they are not empirical factors that need to be adjusted to match model calculations to experimental data.

Except for two empirically justified phenomenological assumptions the results presented above are a mathematical consequence of the scaling that led to the diagnostic relationship: $-\gamma d = p_{,t} + v_p p_{,p}$. The phenomenological assumptions invoked are (i) quasi-normal behaviour of the large scales and (ii) the Kolmogorov relationship. With these qualifications in mind the expressions derived are mathematically exact. These results must be understood as leading-order contributions to the dilatational covariances. Higher-order terms scale with the anisotropy, b_{ij}.

4. Implications of the results

4.1. *The local dilatational dissipation*

Immediately apparent, in the light of other models for the compressible dissipation, is its dependence on mean flow gradients, the Reynolds number and the Kolmogorov scaling coefficient. The Mach number dependence, M_t^4 is stronger than the M_t^2 dependence in the Sarkar model for the dilatational dissipation, though less steep, for small M_t, than the exponential dependence of Zeman's (1993) model. (It should be kept in mind, as will be discussed shortly and in §5, that Sarkar treats a different problem.)

Reynolds number dependence of the dilatational dissipation

While the dilatational dissipation has been expressed in terms of M_t and R_t, it is useful to consider the fact that $M_t^4/R_t = \nu\varepsilon_s/c^4$ which implies that

$$\varepsilon_c \sim \frac{\nu}{c^4}\varepsilon_s^2, \qquad \varepsilon_c \sim \frac{\nu}{c^4}\left[\hat{S}^2 + \hat{S}^4\right]\varepsilon_s^2. \tag{90}$$

It is seen that the local dilatational dissipation's dependence on the turbulent Mach number is through its dependence on temperature and not kinetic energy. In Blaisdell's homogeneous shear DNS the quantity $\chi_\varepsilon = \varepsilon_c/(\varepsilon_s + \varepsilon_c)$ becomes independent of M_t (for $M_t > 0.15$) even though M_t continues to grow; the above scaling is consistent with this feature. It should be noted that the substitution $\varepsilon \sim \tilde{u}^3/\ell$ recovers a M_t dependence.

The analysis has produced a representation for the local dilatational dissipation that depends on Reynolds number. The magnitude of the dilatational dissipation depends on the viscosity: for a fixed M_t, the local dilatational dissipation vanishes as $R_t \to \infty$. For $M_t^2 < 1$ it appears that the usual interpretation of dissipation quantities as spectral fluxes is not appropriate. That the compressible dissipation might not be interpreted as a spectral flux (as is the solenoidal dissipation) is suggested by results given in the EDQNM of Bataille (1994). In Bataille's (1994) simulation the solenoidal spectrum, E_{ss}, is found to scale as the usual $\kappa^{-5/3}$; the compressible spectrum, for small Mach number, is much steeper and scales as $E_{cc} \sim \kappa^{-11/3}$. Multiplying by κ^2, the solenoidal and dilatational dissipations are found to scale as $\kappa^{1/3}$ and $\kappa^{-5/3}$, respectively. The negative power law scaling of the local dilatational dissipation indicates that, unlike the enstrophy, the dilatation is smaller at larger wavenumber.

Consider an approximate spectral Mach number, $M_t^2(\kappa) \sim E(\kappa)\kappa/c^2$. Using the incompressible spectrum (as the compressible spectrum falls off faster) produces $M_t\ (\kappa) \sim \kappa^{-1/3}$. This suggests that the local dilatational dissipation is a result of a competition between effects that are important at different scales: the energy in the fluctuating dilatation at the larger scales and the sharp gradients necessary for viscous dissipation at the small scales. This is consistent with the idea that, for fixed M_t, increasing R_t by decreasing the viscosity adds more small scales to the field that are also more divergence free. The length scales at which the velocity gradients are large enough to undergo viscous dissipation become smaller and simultaneously more divergence free; a net reduction of the dilatational dissipation results.

This effect is not expected to be seen in current DNS turbulence in which there is no unambiguous spectral gap: the small scales of the motion are almost as compressible as the large scales of the motion. For the compressible homogeneous DNS typically $R_t = 4k^2/9\varepsilon\langle\nu\rangle < 100$. A spectral gap is typically not seen until $R_t > 10^3$. Attention might be drawn to the results of the numerical experiments of Blaisdell & Zeman (1992) in which the dilatational dissipation was found to be associated with what are identified as large-scale acoustic waves. Note also the scalings for the compressible dissipation recently observed in LES by Shao, Fauchet & Bertogio (1996) at Lyon: they have observed that $\varepsilon_c \sim M_t^4/R_t$, as is derived here and in Ristorcelli (1995).

Gradient Mach number

Sarkar (1995) has found a dependence of the effects of compressibility on what is called a gradient Mach number: $M_g = S\ell/c$. This dependence is not new; it has been noted in the RDT analyses of Durbin & Zeman (1992), Cambon *et al.* (1993) and Simone *et al.* (1997). The gradient Mach number can be thought of as a local convective Mach number, $M_c = (U_1 - U_2)/(a_1 + a_2)$, as is typically used to parameterize the compressible mixing layer.

For an arbitrary three-dimensional flow the dependence on the gradient Mach number can be replaced with a mean strain (or distortion, Simone *et al.* 1997) Mach number and $R^2 = W^2/S^2$. A mean strain Mach number can be defined as $S\ell/c = \frac{2}{3}\alpha(Sk/\varepsilon_s)M_t \simeq \frac{2}{3}\hat{S}M_t = M_S$. The strain Mach numbers highlight the sensitivity that compressible flows have to velocity gradients as manifest in the so-called rapid portion of the dilatational covariances,

$$\varepsilon_c^r \sim \frac{M_t^2 M_S^2}{R_t}\,\varepsilon_s\,[3+5R^2]\left[\tfrac{3}{5}I_3^r + \left(\tfrac{1}{15}\hat{S}\right)^2[13+15R^2]\,\alpha^2\,I_1^r\right]; \tag{91}$$

for a sheared flow the dilatational dissipation is proportional to the gradient Mach number to the second and fourth powers. Here $R^2 = W^2/R^2$.

Though Sarkar's (1995) subject is the changes in the anisotropy of the turbulence due to compressibility his arguments are applicable to both the dilatational covariances. Sarkar's (1995) arguments indicate that the effects of compressibility are much larger in the mixing layer than in the equilibrium boundary layer: the mixing layer is stabilized with respect to the boundary layer by compressibility. The difference between the compressible mixing layer and the boundary layer flow is parameterized by the gradient Mach number, M_g. In Sarkar's (1995) examples, M_g (proportional to M_S) for the mixing layer can be an order of magnitude larger than that for the boundary layer. The same reasoning using the mean gradient Mach number applied to the dilatational dissipation indicates that compressibility dissipation effects are substantially more important for the mixing layer than for the wall layer. Using Sarkar's (1995) values and definition of the gradient Mach numbers, $M_g \sim 6$ in his example of a mixing layer while in the boundary layer $M_g \sim 1$ and the effects of the compressible dilatation are an order 6^2 more important in the mixing layer than in the boundary layer.

The importance of the dilatational terms is difficult to assess *a priori*; scaling, as they do, with M_t^4 and R_t^{-1} suggests that they are negligible. This is certainly true for the slow compressible dissipation; and is likely to be the case for the rapid portion. There is a possibility that for say $Sk/\varepsilon \gg 10$ the dilatational dissipation's dependence on M_t^4 might be compensated for by the $(Sk/\varepsilon_s)^4$ behaviour. In general the analysis shows for typical $Sk/\varepsilon \sim 6$ that $\varepsilon_c^r \gg \varepsilon_c^s$ and both are small. This is consistent with recent results of Vreman *et al.* (1996) and Simone *et al.* (1997). This is not the case for the homogeneous shear of Blaisdell (1996). In Blaisdell's flow the dilatational dissipation is in the range of 5–10% of the solenoidal dissipation. The Reynolds number is low and the M_t^4 dependence may well be compensated for by the dependence on $(Sk/\varepsilon_s)^4$.

4.2. *The pressure–dilatation*

In this subsection the results for the pressure–dilatation are discussed. Comparisons to the physical behaviour seen in diverse prototypical flows are made.

Increased inertia

The role of the pressure–dilatation, as an agency of transfer between internal and kinetic modes of energy, appears to have first been noticed by Zeman (1991) and explored further in homogeneous shear DNS by Sarkar *et al.* (1991*a*). With a simple rearrangement, the pressure–dilatation expressions can be seen to increase the turbulence's inertia. Earlier the pressure–dilatation covariance was written in terms of the advective derivative. By expanding those differentials the expression for $\langle pd \rangle$ can be rearranged in the $\overset{\circ}{k}$ equation to produce terms representing an 'added mass' effect and two additional source terms:

$$[1 + I_{pd} M_t^2]\ \bar{\rho}\ \frac{\mathrm{D}}{\mathrm{D}t}\ k = \bar{\rho} P_k - \bar{\rho}\varepsilon - k\ I_{pd}\ \frac{\mathrm{D}}{\mathrm{D}t}\ [\bar{\rho} M_t^2] - \bar{\rho} k\ M_t^2 I_{pd}^r\ \frac{\mathrm{D}}{\mathrm{D}t}\ \mathscr{T}.$$

As I_{pd} scales with $(Sk/\varepsilon_s)^2$ the added mass effect scales with the square of the gradient Mach number $M_g^2 \sim M_t^2 (Sk/\varepsilon_s)^2$. The faster the turbulence is sheared the more work it takes to change the energy of a mean fluid particle.

The isotropic decay

The pressure–dilatation covariance in isotropic (decaying) turbulence is

$$\langle pd \rangle = \chi_{pd} M_t^2\ \varepsilon + O(M_t^4). \qquad (92)$$

The pressure–dilatation is *positive* indicating a net transfer of energy from the mean temperature to the turbulence. The $\overset{\circ}{T}$ and $\overset{\circ}{k}$ equations for this case are written

$$-\ c_v \frac{\mathrm{D}}{\mathrm{D}t} T = \frac{\mathrm{D}}{\mathrm{D}t} k = -(1 - \chi_{pd} M_t^2)\varepsilon. \qquad (93)$$

The factor multiplying the dissipation is always positive, $1 - \chi_{pd} M_t^2 > 0$. The pressure–dilatation will act to *slow* the rate of decrease of k by shunting energy stored in the mean temperature to the kinetic energy of the turbulence. Sarkar *et al.* (1991*b*) have assumed $(-\langle pd \rangle + \varepsilon_c = \alpha_1 M_t^2 \varepsilon_s)$ and the turbulence energy equation can be rewritten

$$\frac{\mathrm{D}}{\mathrm{D}t}\ k = \langle pd \rangle - \bar{\rho}\varepsilon_s - \bar{\rho}\varepsilon_c = -(1 + \alpha_1 M_t^2)\varepsilon_s. \qquad (94)$$

The results of the present analysis and Sarkar *et al.* (1991*b*) are not contradictory: they treat two different problems. The present analysis treats turbulence in which compressible effects are generated by the turbulent motions. Sarkar *et al.* (1991*b*) appears to have treated homogeneous *non-compact* turbulence on which is superposed, by the initial conditions, an M_t^2 dilatational wave field. The local dilatational field is an order-M_t^4 effect when the dilational fluctuations are generated, not by the initial conditions, but by the vortical fluctuations.

4.2.1. *Scaling the pressure–dilatation: the isotropic decay*

For the isotropic decay the expression for the pressure–dilatation can be rearranged:

$$\frac{\langle pd \rangle}{M_t^2\ \varepsilon_s} = \chi_{pd} = \frac{\frac{4}{3} I_1^s}{1 + \frac{4}{3} I_1^s M_t^2}. \qquad (95)$$

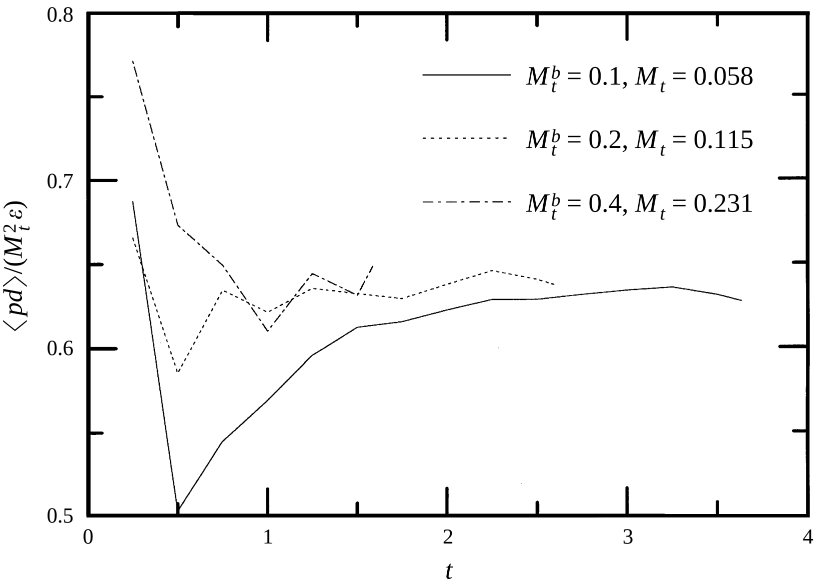

FIGURE 2. Scaled pressure–dilatation in isotropic decaying turbulence.

Terms of order M_t^4 have been dropped. Earlier estimates given in Ristorcelli (1995), shown above, indicate $I_1^s = 0.5 - 0.3$. The theory therefore predicts an asymptote for χ_{pd} as the turbulent Mach number vanishes:

$$\chi_{pd} = \frac{\frac{4}{3}I_1^s}{1 + \frac{4}{3}I_1^s M_t^2} \rightarrow 0.666 - 0.40 \qquad \text{as} \qquad M_t^2 \rightarrow 0. \tag{96}$$

The agreement with the DNS shown in figure 2 is very good – the scaling asymptotes in the range suggested in Ristorcelli (1995) as the initial conditions fade. It appears that the analysis without any *a posteriori* adjustment of constants has been verified. The DNS results, shown in figure 2, were provided by Blaisdell for three different initial turbulent Mach numbers. They are recent compressible DNS reflecting a consistent set of initial conditions described in Ristorcelli & Blaisdell (1997). As a service to the reader the figure includes two definitions of the turbulent Mach number: that used by Blaisdell in his simulations, M_t^b, and that which comes from the perturbation theory given here.

The pressure–dilatation in homogeneous shear

The perturbation expansion constructed earlier allows an interesting connection with the results of Sarkar (1992). In primitive variables $\langle pd \rangle = \langle (p_1 + \epsilon^2 p_2 + ...)(d + \epsilon^2 ....)\rangle = \langle p_1 d \rangle + O(M_t^2)$ is a statement that the primary (averaged) contribution to the pressure–dilatation is from the incompressible pressure, p_1. This is consistent with Sarkar's (1992) homogeneous shear DNS. He identifies compressible and incompressible pressure fields and compares their contributions to the time-averaged pressure–dilatation. He finds quite conclusively that the most important cumulative contribution to the pressure–dilatation is made by the incompressible pressure.

Consider the following lowest-order simplification of $\langle pd \rangle$ (assuming $\overset{\circ}{\mathscr{T}} \simeq 0$ the pressure–dilatation can be written $\langle pd \rangle = -\chi_{pd} M_t^2 \,[P_k - \varepsilon - \frac{3}{4} M_t^2 \gamma(\gamma - 1)\varepsilon]$):

$$\langle pd \rangle = -\chi_{pd} M_t^2 \,[P_k - \varepsilon]. \tag{97}$$

Several items are worth noting. The pressure–dilatation will only alter the turbulence in situations when $P_k \neq \varepsilon$. The effect will be controlled by M_t^2 and $(Sk/\varepsilon)^2$ (see the definition of χ_{pd} in the Appendix). The non-dimensional quantity $M_t^2(Sk/\varepsilon_s)^2$ has been related to a gradient Mach number; the larger the gradient Mach number the larger the pressure–dilatation effect (if $P_k \neq \varepsilon$). It should noted that Vreman *et al.* (1996) find pressure–dilatation negligible and for good reason: their flow is stationary. The pressure dilatation is zero for stationary flows in which $P_k = \varepsilon$ or equivalently $(\mathrm{D}/\mathrm{D}t)k = 0$, (see equation (17) of Vreman *et al.* 1996).

Also worth noting is the change of sign of $\langle pd \rangle$ noted by Sarkar *et al.* (1991*a*) and by Blaisdell *et al.* (1991). For flows with small turbulence production, $P_k < \varepsilon$, the pressure–dilatation is positive and 'destabilizing'. Clearly, if the production exceeds the dissipation the pressure–dilatation covariance is negative and 'stabilizing" and a net transfer of energy from the turbulence field to the mean internal energy occurs. The pressure–dilatation can be either stabilizing or destabilizing and this effect is amplified by the gradient Mach number. A similar phenomena is seen in the results of Simone *et al.* (1997) in the context of their discussion on the stabilizing or destabilizing effects of the gradient Mach number on b_{12}. Their simulations start with isotropic initial conditions for which, of course, the production is zero. The short-time results for their DNS indicate that larger gradient Mach numbers are associated with larger values of b_{12}, i.e. the gradient Mach number is destabilizing. As the simulation continues further the larger gradient Mach number simulations become stabilized with respect to the smaller gradient Mach number flows, thus Sarkar's (1995) conclusion follows. This is very similar to the behaviour seen in the pseudo-sound expressions for $\langle pd \rangle$ where the stabilizing/destabilizing effect is determined by the relative size of P_k/ε. One might speculate that the different behaviours in the Simone *et al.* (1997) simulation might be correlated with the sign of $P_k/\varepsilon - 1$.

The critical M_{tc}^2, as a function of shear rate, anisotropy and ratio of specific heats, at which the pressure–dilatation changes sign is found by setting $\langle pd \rangle = 0$. To lowest order

$$M_{tc}^2 = \frac{4}{3} \frac{\frac{1}{2}\, b_{12}\, \hat{S} - 1}{\gamma(\gamma - 1)}. \tag{98}$$

This approximation is only valid in flows that have achieved, consistent with the analysis, some sort of structural equilibrium: this does not occur until $St > 10$ in homogeneous shear DNS starting from isotropic initial conditions.

Scalings for the pressure–dilatation in homogeneous shear

The scalings predicted by the analytical results are now compared to recent DNS of Blaisdell (1996, personal communication). Taking into account the definition of the coefficients in the expression for the pressure–dilatation it is seen that

$$\langle pd \rangle \sim -\alpha^2 \left(\frac{Sk}{\varepsilon_s}\right)^2 M_t^2\, \varepsilon_s I_1^r \left[\frac{P_k}{\varepsilon_s} - 1\right] \left[1 + \frac{1}{P_k/\varepsilon_s - 1} \frac{\mathrm{D}}{\mathrm{D}(St)} \frac{Sk}{\varepsilon_s}\right]. \tag{99}$$

The appropriate scaled integral of $\langle pd \rangle$ will be taken. The integrals are taking following Sarkar (1992) who showed that the major contribution to the time average of $\langle pd \rangle$ is due to the incompressible pressure; oscillatory compressible pressure fluctuations make little contribution to time averages. The integrand will be weighted by quantities

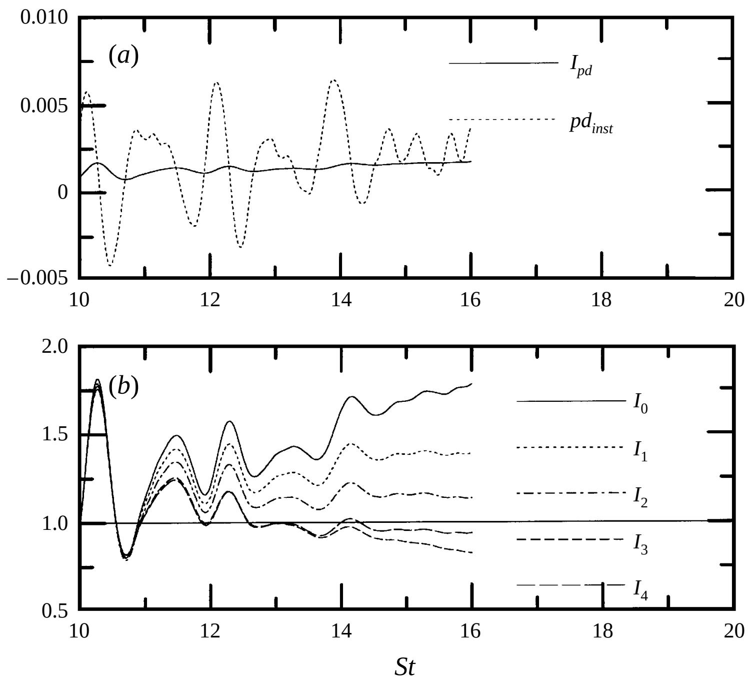

FIGURE 3. Scaled pressure–dilatation for 'equilibrium' homogeneous shear.

calculated by the DNS suggested in the expression above:

$$I_0 = \int \langle pd \rangle, \quad I_1 = \int \frac{\langle pd \rangle}{\varepsilon_s}, \quad I_2 = \int \frac{\langle pd \rangle}{\varepsilon_s[P_k/\varepsilon - 1]},$$
$$I_3 = \int \frac{\langle pd \rangle}{M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2}, \quad I_4 = \int \frac{\langle pd \rangle}{\alpha^2 M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2}. \tag{100}$$

The symbol $\int$ is used to denote the operation $(1/St)\int(\ )d(St)$. Figure 3(a) shows the instantaneous and time-averaged values of $\langle pd \rangle$. Figure 3(b) shows the scaled integrals of $\langle pd \rangle$ given above. The integrations shown have been taken during the latter portions of the DNS, as this is the portion of the flow during which a semblence of a structural equilibrium is approached. This is assessed by the approach of I_1^r to its final value: for the last four dimensionless times I_1^r is within 25% of its final value. It is also indicated by the substantial diminishing of the contribution of $(\mathrm{D}/\mathrm{D}t)\hat{S}$. The averaging procedure was started at $St = 9$ and all values are normalized by their value at $St = 10$. The period $St = 10$ to $St = 16$ is equivalent to little less than one and a half eddy turnovers.

It is seen that the scaling, in I_3, $M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2$, featuring as it does the gradient Mach number, exceeds the collapse expected. It had been originally thought that the quantity $M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2$ would collapse the data – the Kolmogorov coefficient had been expected to be approximately constant as is seen in many flows (Sreenivasan 1995; Yeung & Zhou 1997). This is not the case for the Blaisdell DNS data – homogeneous shears are non-equilibrium flows. The Kolmogorov coefficient drops substantially towards the end of the simulation to what may be considered its 'typical' value, $\alpha \approx 1$. When the Kolmogorov scaling coefficient, taken from the DNS, is accounted for the collapse is very good. This point is made to emphasize

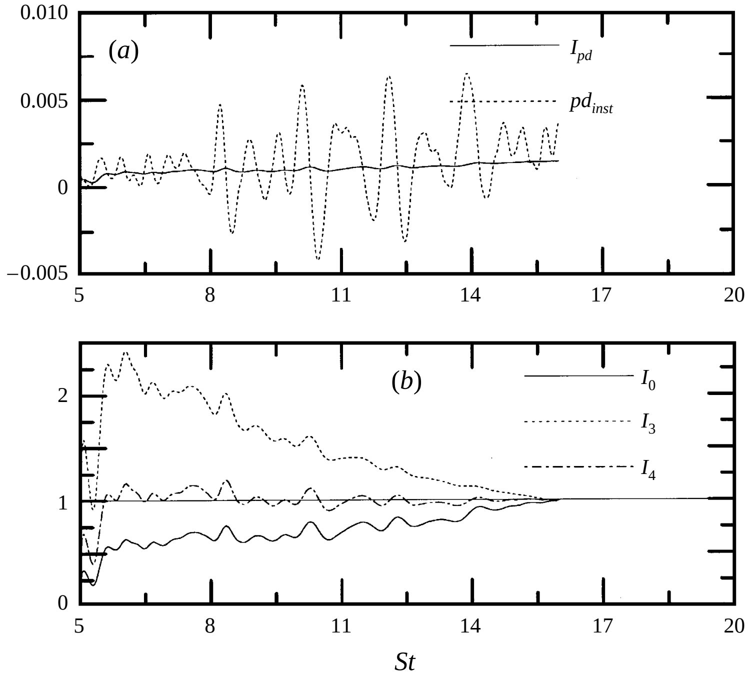

FIGURE 4. Scaled pressure–dilatation for 'non-equilibrium' homogeneous shear.

the inadequacy of the present data on the Kolmogorov coefficient for more complex flows. It is also, perhaps, an adumbration of the potential role this quantity may play in compressible turbulence.

The scalings were also checked during the earlier non-equilibrium portion of the flow, where I_1^r is not constant; its value is computed from the DNS data. This calculation, shown in figure 4, was performed starting at $St = 5$, to allow some consequences of the isotropic initial condition to dissipate and to ensure that $P_k/\varepsilon > 1$ to avoid the singularity in the denominator. Here

$$I_0 = \int \langle pd \rangle, \quad I_3 = \int \frac{\langle pd \rangle}{M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2}, \quad I_4 = \int \frac{\langle pd \rangle}{\alpha^2 I_1^r M_t^2 \varepsilon_s [P_k/\varepsilon - 1](Sk/\varepsilon)^2}. \tag{101}$$

The collapse of the data is substantial over such a large time span. It is possible to write considerably more on these comparisons with DNS. These results are treated in more detail in Ristorcelli (1997).

5. Discussion and clarification of limitations and assumptions

In order to make the mathematical development possible it has been necessary to make simplifying assumptions. This section is a compilation of the assumptions used; the complexity of the analysis may have obscured the reasonableness or inadequacy of the assumptions. The section is also included in order to ensure that the applications of these representations be made with an awareness of their limitations. It is hoped that discussion of the assumptions will suggest future work to account for the potential shortcomings of this one.

Low turbulent Mach number

The square of the turbulent Mach number, $M_t^2 = \frac{2}{3}k/c^2$, has been assumed to be small. This is the case for many supersonic flows of engineering interest. The low *turbulent* Mach number squared assumption should not be understood to imply a low *mean* flow Mach number. It is a statement quantifying the modest compressibility of the fluctuating velocity field.

While a low-M_t^2 analysis is expected to be appropriate for most flows of aerodynamic interest, it must be noted that the nonlinear self-interaction terms of the compressible velocity field, necessary for a compressible cascade, are absent in this analysis (they are negligible). As $M_t \to 1$ nonlinear terms and shocklets are expected to become important. The present article has no relevance to flows in which shocklets are a major portion of the dissipation. Bataille's (1994) EDQNM results show that as M_t increases the slope of E_{cc} decreases; as $M_t \to 1$, the slope of E_{cc} approaches the slope of E_{ss} indicating the existence of a compressible cascade and the dilatational dissipation has positive exponent in wave space. In that limit the compressible dissipation is not expected to be a function of Reynolds number. A compressible nonlinear cascade is not expected to be important in most supersonic flows and cannot be accounted for in this analysis. On this point consider the homogeneous shear in Blaisdell *et al.* (1991, 1993) discussed in Blaisdell & Zeman (1992): shocklets, for moderate turbulent Mach numbers, contribute very little to the dilatational dissipation.

Compact source

The low-M_t assumption is equivalent to the *compact source* assumption of aeroacoustics. In acoustics there is a source compactness parameter: $\omega_c \ell / c$, where ω_c is a characteristic frequency of the acoustic radiation. Applying this measure of compactness to the present situation, it is found that the source compactness is proportional to the turbulent Mach number, $\omega_c \ell / c \sim \varepsilon \ell / kc \sim M_t$ – the correlation length scale of the flow structures producing the dilatational field is smaller than the wavelength of the propagating field.

Compact flow

The scalings employed imply that the source of compressibility at any point in the turbulence is due to the turbulence within an integral scale of that location. The compressible effects do not result from any externally imposed 'acoustical' fields, or radiation from far-field turbulence; nor are they an adjustment of the flow to initial conditions with an arbitrary compressible component not generated by the turbulence. This combined problem has been investigated in Durbin & Zeman (1992).

Homogeneous compressible DNS is not a compact flow. Homogeneous DNS correspond to turbulence immersed in a general background random wave field due to inconsistent initial conditions as well as non-local contributions to the variances of propagating fields such as $\langle dd \rangle$. The model problem treated by Sarkar *et al.* (1991*b*) appears to have addressed turbulence of scale ℓ irradiated by an infinite external acoustic field generated by turbulence whose statistics are the same as those of the local turbulent region.

Kolmogorov scaling

The Kolmogorov scaling relation, $\varepsilon = \alpha(\frac{2}{3}k)^{3/2}/\ell$, has been used several times. The scaling has substantial empirical validity in many isotropic incompressible flows with different large-scale forcings (Yeung & Zhou 1997; Sreenivasan 1984). However, in non-ideal flows that are strained, anisotropic or inhomogeneous α cannot be thought of as a universal constant. The scaling $\varepsilon \sim (\frac{2}{3}k)^{3/2}/\ell$ is nonetheless a useful approximation in many high Reynolds number flows (Sreenivasan 1995*a*): this is to be expected as the scaling comes from spectral arguments in ranges of the cascade in which the fluctuating strain is large enough to be reasonably insensitive to the smaller strain associated with the mean deformation. This, of course, is not the case for rapidly distorted flows in which a spectral equilibrium, the assumption that underlies the Kolmogorov scaling, will not exist. In such rapidly distorted flows the mean strain is higher than the fluctuating strain for a sizable portion of the inertial subrange. In most high Reynolds number flows of the transversely sheared type, nonlinearities develop rapidly and the turbulence adjusts to imposed strains rapidly producing more nominal values of the relative strain parameter, Sk/ε_s; only a small portion of the flow domain corresponds to a flow situation in which a scaling of the type $\varepsilon \sim \tilde{u}^3/\ell$ can be argued to be inadequate.

Sreenivasan (1995*a*) has assessed the accuracy of the Kolmogorov scaling relation in several canonical (incompressible) simple shear flows. For homogeneous shear the data indicate $\alpha \sim 1$–2. For the log layer or wake flows $\alpha \sim 4$. The crucial point is that α is a *flow-specific quantity*; even in the homogeneous DNS α can vary significantly during the course of the simulation (Rogers, Moin & Reynolds 1986; Blaisdell 1996, personal communication). There appears to be very little known about the scaling coefficient in non-ideal flows. Some sort of parameterization for flows in structural equilibrium appears possible (Sreenivasan 1995*a*). Nonetheless the coefficient α is a defined quantity, and is measurable in the class of flows of interest. As has been mentioned its appearance, in as much as it links dimensionally and phenomenologically, the energy, the spectral flux and a two-point correlation length scale, is a measurable indication of the dependence on large scales.

Quasi-homogeneity

The assumption of 'quasi-homogeneity' is made throughout the mathematical development: this is an assumption of homogeneity on the scale ℓ which is to say $\ell/L\langle 1$ where L is the scale of the inhomogeneity. One of the reviewers has pointed out that the analysis also presumes quasi-homogeneity *in time* of the mean strain. This comes from the fact that, a unlike an incompressible homogeneous flow, a homogeneous compressible flow does not allow a steady arbitrary strain (except for pure shear): thus a stationary mean assumption violates strict homogeneity for a compressible mean flow. The analysis therefore holds for flows in which (i) the mean flow is homogeneous and quasi-stationary on the time scale of the turbulence $S^{-1}\overset{\circ}{S}\,k/\varepsilon_s < 1$ or (ii) the mean flow is stationary in time but quasi-homogeneous in space. Additional details on the use of homogeneity in compressible DNS can be found in Blaisdell *et al.* (1991, Chapter 2 and Appendix A).

The mean pressure and mean density (and thus sound speed) have been assumed locally constant – constant over a length scale over which the turbulence is correlated. The locally constant mean density assumption implies that the mean bulk dilatation is negligible. For flows in which the mean density and mean pressure vary appreciably over an integral scale the present representation captures only a portion of the physics.

In the constitutive relation for the pressure dilatation the flux terms $\langle v_k pp\rangle_{,k}$ have been neglected on the grounds of homogeneity (or because they are triple moments). It should be observed that where the flux terms are important typically coincides with regions in which production is not important and thus M_t^2 is small.

Isotropy

All the expressions presented have been obtained assuming that the leading-order contribution to various two-point integrals, which are integrals over ellipsoids, can be approximated by integrals over a sphere of the same volume. The problem is otherwise, without resorting to empiricism, intractable. The resulting expressions are thus leading-order terms in a series expansion in powers of the anisotropy of the turbulence. The substance of such a procedure for those unfamiliar with such a style of thinking can most easily be seen in Shih, Reynolds & Mansour (1990) where it is carried out in the Fourier domain or in Lumley (1970).

Higher-order terms allowing contributions from the anisotropy are straightforward in concept but intractable without empiricism. For *unstrained* flows their contributions will be nominal, on the order of the scalar $M_t^2\,\|b^2\|$, since $\langle pd\rangle$ is scalar. In homogeneous shear flows $\frac{1}{2}\|b^2\| \sim 0.058$ is also small. This argument is however, not appropriate for strained flows; there will be second-order terms, $\boldsymbol{b}(\nabla \boldsymbol{V})^2$, which, it can be argued, should be small. Some amplification of this point, using the rapid portion of the pressure–dilatation, is now given. Consider the leading order contribution to the pressure variance derived above: $\langle pp\rangle^r = \frac{1}{15}\rho_\infty^2 \frac{2}{3}k\ell^2\ [3S^2 + 5W^2]\ I_1^r$. The next higher-order terms, in a small-anisotropy expansion, contribute terms of the form $V_{i,j}\,V_{p,q}\,b_{jq}$. Expressing the velocity gradient in terms of strain and rotation produces an expression of the form

$$\langle pp\rangle^r \sim\ C_0[3S^2 + 5W^2] + C_1 b_{ij}[S_{ij}^2 + W_{ij}^2] + C_2 b_{ij}[S_{ik}W_{kj} + S_{jk}W_{ki}]. \tag{102}$$

Such a form has also been given in Durbin & Zeman (1992) using the usual tensor representation arguments. They obtained values for C_0 and C_1 appropriate for the small-time behaviour of rapidly distorted flows. In flows for which nonlinear effects play a primary roll, as they will in any flow that evolves an eddy turnover time or so away from its initial condition, the present procedure gives a value for C_0. The procedure cannot predict values for C_1 or C_2 without empiricism. Certainly calibration can produce constant values for these quantities. For simple shear flows $W \sim S$ and the scaling of higher-order terms is also S^2 – thus the successful collapse of the DNS data. For highly anisotropic flows with rapidly changing b_{ij} the scaling will be less exact and this issue may have to be investigated more closely in order to construct. Given the modest contributions of $\langle pd\rangle$ and the small contribution of ε_c these issues do not appear to warrant further attention. This generalization, however, must be qualified: there are a wide variety special strategies to enhance mixing and these effects may not be negligible in a wider class of flows.

Quasi-normality

The assumption of quasi-normality for the large scales of the flow has been made to achieve closure. The quasi-normal assumption for the dynamically significant large scales of a flow is an exceedingly good approximation in both homogeneous and inhomogeneous flows. The adequacy of the quasi-normal assumption has been investigated for several decades now. Batchelor (1951, 1953) appears to have been the first to have presented experimental evidence of its adequacy when invoked with respect to the large scales of the flow. A spectral version of this assumption is used in the EDQNM theory which since its inception, as presented in Orszag (1970), has produced useful results. The adequacy of the assumption for the large scales of the flow has been documented in several experimental works – McComb (1990) gives a summary of these results. Even in inhomogeneous compressible flows, Elliot & Samimy (1990) have found that the quasi-normal assumption is valid in the dynamically significant central portions of the mixing layer where the turbulent Mach number is largest.

In the quasi-normal approximation, the term $\langle \dot{p} v_q p_{,q} \rangle$ has been dropped from the constitutive relation for the variance of the dilatation. In as much as the dilatational variances are a small contribution to the overall energy levels of weakly compressible flows, this term has not been retained. The more important pressure–dilatation term does not suffer from any such approximation.

Isentropy

The turbulent fluctuations that contribute to the dilatational covariances have been assumed to obey the adiabatic gas law. This uncouples the problem from non-isentropic aspects of a compressible flow which are expected to be important in wall-bounded flows under non-adiabatic conditions.

Viscous effects

In some of the manipulations involving the acceleration, the quantity $(-p_{,i} - v_k v_{i,k})$ has been used to replace the acceleration, $\overset{\circ}{v}_i$ – viscous effects are neglected. In general one would expect viscous effects on the acceleration to be important. This would be expected if it were the small scales that contributed to the dilatation in low-M_t flows – in the same way it is the small scales that contribute to the enstrophy. However, in the absence of shocklets, viscous effects on the scales of the motion that contribute to the dilatation in higher Reynolds number flows are neglected.

Initial conditions

In compressible DNS appropriate initial conditions are required: the leading-order low Mach number expansion indicates that the density and temperature variances are related to the incompressible field according to $\gamma\langle\rho\rho\rangle = \gamma/(\gamma-1)\langle\theta\theta\rangle = \langle pp\rangle$. The so called 'incompressible' initial conditions $\langle\theta\theta\rangle = \langle\rho\rho\rangle = 0$ are *inconsistent* with finite initial M_t. Whether this is important or not and how long its effects will last before being obscured by the coupling between the dilatational and vortical fields (Blaisdell, Mansour & Reynolds 1993), is sure to depend on the particular situation. These speculations about the appropriateness of initial conditions (Ristorcelli 1995), and their impact on interpretation of DNS data have begun to be substantiated in Ristorcelli & Blaisdell (1997). The initial condition transients in the compressible DNS are analogous to the transients associated with the free oscillations of an underdamped linear system, $y'' + Re_c^{-1} y' + \omega_0^2 y = 0$, relaxing from its initial condition. This situation appears to correspond to the analysis followed by Erlebacher *et al.* (1990) and Sarkar *et al.* (1991*b*). In this article the treatment for the effects of compressibility can be thought of as analogous to the forced oscillator: $y''+Re_c^{-1}y'+\omega_0^2 y = f(\omega_t)$. The forcing comes from the energy-containing turbulent motions on the time scale $\omega_t \sim \ell/\tilde{u}$ and the dependence on the initial conditions is assumed to have faded.

6. Summary and conclusions

A small $M_t^2 = \frac{2}{3}k/c^2$ perturbation procedure has produced a *diagnostic* constitutive relationship, $-\gamma d = p_{,t} + v_k p_{,k}$, relating the fluctuating dilatation to the pressure field associated with the solenoidal portions of the velocity field. Moments of the diagnostic relation then produce constitutive relations for the dilatational covariances.

Application of the methods of statistical fluid mechanics and the assumptions of quasi-homogeneity and retaining only leading-order isotropic portions of the two-point integrals produces expressions for the covariances with the fluctuating dilatation. Except for the well-established Kolmogorov scaling, $\ell \sim (\frac{2}{3}k)^{3/2}/\varepsilon_s$, and the quasi-normal assumption, no additional phenomenological assumptions are made. The analysis is, in the low-M_t^2 limit, exact and produces representations for the effects of compressibility in which there are no undefined constants.

The present analysis treats only the 'scalar' effects of compressibility – the reduction in k through the dilatational covariances in the energy budget; it cannot account for the reduction in the shear anisotropy, b_{12}, or the normal anisotropy, b_{22}, so important to the production mechanism for the shear stress, $\langle v_1 v_2 \rangle$. To account for these more substantial structural effects appears to require a compressible pressure–strain representation accounting for the effects of compressibility. This has been indicated in Blaisdell & Sarkar (1992), Vreman *et al.* (1996), Simone *et al.* (1997).

This article has focused primarily on producing and understanding the mathematical consequences of a few assumptions associated with small M_t^2. As a perturbative procedure it will express the effects of compressibility in term of the underlying solenoidal velocity field about which much is known. The expressions obtained may be viewed as the leading-order term in a more general expression in which successive terms scale with the anisotropy and inhomogeneity of the flow. The comparisons with the DNS are excellent and show an excellent collapse of the data without use of empirical constants. While a comprehensive validation of the present analysis is not the subject of this article, see Ristorcelli (1997), the procedure has been corroborated by the DNS and now more difficult structural aspects of compressibility might be addressed.

Some findings are now summarized:

(i) The pressure–dilatation is found to be a non-equilibrium phenomena. It scales as $M_t^2(Sk/\varepsilon_s)^2[P_k/\varepsilon_s - 1]$. For it to be important requires *both* the square of the gradient Mach number, M_S^2, to be non-negligible *and* $P_k \neq \varepsilon$. In as much as the pressure–dilatation can be either positive or negative its dependence on the gradient Mach number indicates that the gradient Mach number can be either stabilizing or destabilizing. These predictions are consistent with the DNS of Simone *et al.* (1997) who observe such behaviour as related to the anisotropy, b_{12}.

(ii) Both the dilatational covariances are functions of the Kolmogorov scaling coefficient; this is expected to be an important feature in models for compressible flows. The Kolmogorov coefficient is a flow-dependent quantity: there is little known about its dependence in non-ideal – anisotropic, strained, inhomogeneous – flow situations. The appearance of the Kolmogorov coefficient, in as much as it links the energy, the spectral flux and a two-point length scale, is an indication of dependence on large-scale structure.

(iii) For high-R_t, low-M_t^2, non-equilibrium flows the dilatational dissipation is found to be less important than the pressure–dilatation. It is found to be a function of the relative strain rate and the solenoidal dissipation scaling as $M_t^4(Sk/\varepsilon_s)^4 R_t^{-1}$.

(iv) The dilatation dissipation is dependent on the viscosity: for fixed M_t, as $R_t \to \infty$, the local dilatational dissipation vanishes. The compressible dilatation cannot be understood as a spectral cascade rate set by the large scales. The dependence on the Reynolds number suggests that assessing the importance of the dilatational dissipation on the basis of low Reynolds number numerical simulations may be misleading when applied to higher Reynolds number flows.

(v) The analysis also suggests results relevant to initial conditions used in the DNS of compressible turbulence. It is seen that a finite turbulent Mach number implies a finite compressible component of the turbulence field. Compressible numerical simulations starting from 'incompressible' initial conditions with finite Mach number are more consistently initialized with finite initial density, temperature and dilatational velocity fields: $\gamma\langle\rho\rho\rangle = \gamma/(\gamma-1)\langle\theta\theta\rangle = \langle pp\rangle$, Ristorcelli & Blaisdell (1997). It appears that initial conditions inconsistent with the finite non-zero turbulent Mach number associated with the incompressible field will create a spurious wave field (Ristorcelli & Blaisdell 1997), that will attenuate slowly.

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Appendix. Synopsis of the dilatational covariance representations

The analytical results are briefly summarized. The representations for the effects of the compressible dissipation are given by the sum of the slow and rapid portions, $\varepsilon_c = \varepsilon_c^r + \varepsilon_c^s$:

$$\left.\begin{aligned} \varepsilon_c^s &= \frac{16}{3\alpha^2}\frac{M_t^4}{R_t}\,\varepsilon_s\,[I_2^s + 6I_1^sI_3^s],\\ \varepsilon_c^r &= (\tfrac{2}{3})^5\,\frac{M_t^4}{R_t}\,\varepsilon_s\,\hat{S}^2[3+5R^2][\tfrac{3}{5}I_3^r + (\tfrac{1}{15})^2\hat{S}^2[13+15R^2]\,\alpha^2\,I_1^r], \end{aligned}\right\} \qquad \text{(A 1)}$$

where $R^2 = W^2/S^2$ is the mean rotation to strain ratio; $R = 1$ for a pure shear; M_t is the turbulent Mach number, $M_t^2 = \frac{2}{3}k/c^2$, where $c_\infty^2 = \gamma P/\bar{\rho}$ is the local sound speed. The turbulent Reynolds number is given by $R_t = \tilde{u}\ell/\nu = 4k^2/9\varepsilon\nu$ using the facts that $\tilde{u} = 2k/3$ and $\varepsilon_s \sim \tilde{u}^3/\ell$. In general $\ell = \alpha(2k/3)^{3/2}/\varepsilon_s$. The Kolmogorov scaling coefficient, α, is known to be a *flow-specific quantity*. Note that in the definition the characteristic velocity $(\frac{2}{3}k)^{1/2}$ is used. The non-dimensional strain and rotation rates are given by $\hat{S}^2 = (Sk/\varepsilon_s)^2$, $\hat{W}^2 = (Wk/\varepsilon_s)^2$ where $S = (S_{ij}S_{ij})^2$ and $W = (W_{ij}W_{ij})^2$. The strain and rotation tensors are defined in analogy with the incompressible case, i.e. traceless $S_{ij} = \frac{1}{2}[U_{i,j} + U_{j,i} - \frac{2}{3}D\delta_{ij}]$, $W_{ij} = \frac{1}{2}[U_{i,j} - U_{j,i}]$.

The full pressure–dilatation covariance is

$$\left.\begin{aligned} \langle pd\rangle &= -\chi_{pd}M_t^2\,[P_k - \bar{\rho}\varepsilon + T_k - \tfrac{3}{4}M_t^2\gamma(\gamma-1)(P_T + \bar{\rho}\varepsilon + T_T)] - \bar{\rho}k\;\; M_t^2\,\chi_{pd}^r\,\frac{\mathrm{D}}{\mathrm{D}t}\mathscr{T},\\ \chi_{pd} &= \frac{2I_{pd}}{1 + 2I_{pd}M_t^2 + \frac{3}{2}I_{pd}M_t^4\gamma(\gamma-1)},\\ \chi_{pd}^r &= \frac{I_{pd}^r}{1 + 2I_{pd}M_t^2 + \frac{3}{2}I_{pd}M_t^4\gamma(\gamma-1)},\\ I_{pd} &= \tfrac{2}{3}I_1^s + I_{pd}^r\,[3\hat{S}^2 + 5\hat{W}^2], \qquad I_{pd}^r = \tfrac{1}{30}(\tfrac{2}{3})^3\,\alpha^2 I_1^r. \end{aligned}\right\} \qquad \text{(A 2)}$$

Note that $\varepsilon = \varepsilon_s + \varepsilon_c$ and $\mathscr{T} = [3\hat{S}^2 + 5\hat{W}^2]$. The term inside the inner brackets is the right-hand side of the mean temperature equation. The constants, denoted by the I_i, are given by integrals of the longitudinal correlation:

$$I_1^s = \int_0^\infty \xi f'^2 \mathrm{d}\xi, \quad I_2^s = -\int_0^\infty \xi f' \left[f f''' + \frac{4}{\xi} f f'' + \frac{8}{\xi} f' f' - \frac{4}{\xi^2} f f' \right] \mathrm{d}\xi,$$

$$I_3^s = \int_0^\infty \frac{1}{\xi} f'^2 \mathrm{d}\xi, \quad I_1^r = 2\int_0^\infty \xi f \mathrm{d}\xi, \quad I_3^r = -\int_0^\infty \xi^2 f''' + 7\xi f'' + 8 f' \, \mathrm{d}\xi.$$

A quick of order of magnitude estimate for the integrals can be made using $f = \mathrm{e}^{-\xi^2 \pi/4}$. The following values are found: $I_1^s = \frac{1}{2}$, $I_2^s = \frac{41}{27}\pi = 4.77$, $I_3^s = \frac{1}{4}\pi = 0.785$, $I_1^r = 4/\pi = 1.273$, $I_3^r = 3$. The values found from high Reynolds number wind tunnel data are different: $I_1^s = 0.300$, $I_2^s = 13.768$, $I_3^s = 2.623$, $I_1^r = 1.392$, $I_3^r = 3$ (Zhou 1995). The values given for the integrals reflect the assumption of equilibrium isotropic turbulence and are to be understood as suggestive of the order of magnitude that they may have in more complex anisotropic and inhomogeneous situations.

REFERENCES

Abid, R. 1994 On the prediction of equilibrium states in homogeneous compressible turbulence. *NASA Contractor Rep.* 4570. Langley Research Center.

Adumitroaie, V. 1997 Quasi-explicit algebraic modeling of turbulent reacting flows. Dept. MAE, SUNY at Buffalo, Buffalo, NY.

Bataille, F. 1994 Etude d'une turbulence faiblement compressible le cadre d'une modelisation quasi-normale avec amortissement tourbillonnaire. L'Ecole Central de Lyon, France.

Batchelor, G. K. 1951 Pressure fluctuations in isotropic turbulence. *Proc. Camb. Phil. Soc.* **47**, 359.

Batchelor, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

Blaisdell, G. A., Mansour, N. N. & Reynolds, W. C. 1991 Numerical simulations of compressible turbulence. *Rep.* TF-50. MAE, Stanford University, Stanford, CA.

Blaisdell, G. A., Mansour, N. N. & Reynolds, W. C. 1993 Compressibility effects on the growth and structure of homogeneous turbulent shear flow. *J. Fluid Mech.* **256**, 443.

Blaisdell, G. A. & Sarkar, S. 1992 Investigations of the pressure-strain correlation in compressible homogeneous turbulent shear flow. In *Transitional and Turbulent Compressible Flows*. ASME FED 151.

Blaisdell, G. A. & Zeman, O. 1992 Investigation of the dilatational dissipation in compressible homogeneous shear flow. *Proc. Summer Program 1992*. Center for Turbulence Research, Stanford, CA.

Bradshaw, P. 1977 Compressible turbulent shear layers. *Ann. Rev. Fluid Mech.* **9**, 33.

Cambon, C., Coleman, G. N. & Mansour, N. N. 1993 Rapid distortion analysis of compressible homogeneous turbulence at finite Mach number. *J. Fluid Mech.* **257**, 641.

Crow, S. C. 1970 Aerodynamic sound emission as a singular perturbation problem. *Stud. Appl. Maths* **49**, 21.

Durbin, P. A. & Zeman, O. 1992 Rapid distortion theory for homogeneous compressed turbulence. *J. Fluid Mech.* **242**, 349.

Elliott, G. S. & Samimy, M. 1990 Compressibility effects in free shear layers. *Phys. Fluids* A **2**, 1231.

Elliott, G. S., Samimy, M. & Arnette, S. A. 1995 The characteristics and evolution of large scale structures in compressible mixing layers. *Phys. Fluids* **7**, 864.

Erlebacher, G., Hussaini, M. Y., Kreiss, H. O. & Sarkar, S. 1990 The analysis and simulation of compressible turbulence. *Theor. Comput. Fluid Dyn.* **2**, 291.

Ffowcs Williams, J. E. 1969 Hydrodynamic noise. *Ann. Rev. Fluid Mech.* **1**, 197.

Goebel, S. G. & Dutton, J. C. 1991 Experimental study of compressible turbulent mixing layers. *AIAA J.* **29**, 538.

Gutmark, E. J., Schadow, K. C. & Yu, K. H. 1995 Mixing enhancement in supersonic free shear layers. *Ann. Rev. Fluid Mech.* **27**, 375.

Huang, P. G., Coleman, G. & Bradshaw, P. 1995 Compressible turbulent channel flows. *J. Fluid Mech.* **305**, 185.

Karman, T. von & Howarth, L. 1938 On a statistical theory of isotropic turbulence. *Proc. R. Soc. Lond.* A **164**, 192.

Kraichnan, R. H. 1956 Pressure field within homogeneous anisotropic turbulence. *J. Acoust. Soc. Am.* **2**, 64.

Landau, L. D. & Lifshitz, E. M. 1985 *Fluid Mechanics*. Pergamon Press.

Lele, S. K. 1994 Compressibility effects on turbulence. *Ann. Rev. Fluid Mech.* **26**, 211.

Lighthill, M. J. 1952 On sound generated aerodynamically: 1. General theory. *Proc. R. Soc. Lond.* A **211**, 564.

Lumley, J. L. 1970 *Stochastic Tools in Turbulence*. Academic Press.

McComb, W. D. 1990 *The Physics of Fluid Turbulence*. Oxford Science Publications, UK.

Orszag, S. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363.

Papamoschou, D. & Roshko, A. 1988 The compressible turbulent shear layer: an experimental study. *J. Fluid Mech.* **197**, 453.

Proudman, I. 1952 The generation of noise by isotropic turbulence. *Proc. R. Soc. Lond.* A **214**, 119.

Ribner, H. S. 1962 Aerodynamic sound from fluid dilatations. *UTIA Rep.* 68; *AFOSR TN* 3430.

Ristorcelli, J. R. 1995 A pseudo-sound constitutive relationship for the dilatational covariances in compressible turbulence: an analytical theory. *ICASE Rep.* 95-22. NASA Langley Research Center, Hampton, VA.

Ristorcelli, J. R. 1996 Toward a turbulence constitutive relation for rotating flows. *ICASE Rep.* 96-66. (Submitted as part of the proceedings of New Directions in Geophysical Fluid Dynamics and Turbulence *Theor. Comp. Fluid Dyn.* **9**, to appear 1997).

Ristorcelli, J. R. 1997 Pseudo-sound theory for the pressure–dilatation: validation and comparison with other models. *ICASE Rep.* in progress; to be submitted to *Phys. Fluids.*

Ristorcelli, J. R. & Blaisdell, G. A. 1997 Consistent initial conditions for the DNS of compressible turbulence. *Phys. Fluids* **9**, 4.

Ristorcelli, J. R., Lumley, J. L & Abid, R. 1995 A rapid-pressure covariance representation consistent with the Taylor-Proudman theorem materially-frame-indifferent in the 2D limit. *J. Fluid Mech.* **292**, 111.

Rogers, M. M., Moin, P. & Reynolds, W. C. 1986 The structure and modeling of the hydrodynamic and passive scalar fields in homogeneous turbulent shear flow. *Rep.* TF-25. Dept. Mechanical Engineering, Stanford University.

Sarkar, S. 1992 The pressure–dilatation correlation in compressible flows. *Phys. Fluids* A **4**, 2674.

Sarkar, S. 1995 The stabilizing effect of compressibility in turbulent shear flow. *J. Fluid Mech.* **282**, 163.

Sarkar, S., Erlebacher, G. & Hussaini, M. Y. 1991*a* Direct simulation of compressible turbulence in a shear flow. *Theor. Comput. Fluid Dyn.* **2**, 291.

Sarkar, S., Erlebacher, G., Hussaini, M. Y. & Kreiss, H.O. 1991*b* The analysis and modeling of dilatational terms in compressible turbulence. *J. Fluid Mech.* **194**, 15.

Sarkar, S. & Lakshmanan, B. 1991 Application of a Reynolds stress turbulence model to the compressible shear layer. *AIAA J.* **29**, 743.

Shao, L., Fauchet, G. & Bertogio, J.-P. 1996 Asymptotic states of compressible isotropic turbulence. *APS-49th Annual Meeting of the Division of Fluid Duynamic, November, Syracuse, NY.*

Shih, T.-H., Reynolds, W. C. & Mansour, N. N. 1990 A spectrum model for weakly anisotropic turbulence. *Phys. Fluids* A **2**, 1500.

Simone, A., Coleman, G. & Cambon, C. 1997 The effect of compressibility on turbulent shear flow: a rapid-distortion-theory and direct-numerical-simulation study. *J. Fluid Mech.* **330**, 307.

Speziale, C. G., Abid, R. & Mansour, N. N. 1995 Evaluation of Reynolds stress turbulence closures compressible homogeneous shear flow. *Z. Angew. Math. Phys.* **46**, S717.

Spina, E. F., Smits, A. J. & Robinson, S. K. 1994 The physics of supersonic boundary layers. *Ann. Rev. Fluid Mech.* **26**, 287.

Sreenivasan, K. R. 1984 On the scaling of the turbulence energy dissipation rate. *Phys. Fluids* A **27**, 1048.

Sreenivasan, K. R. 1995 The energy dissipation in turbulent shear flows. In *Developments in Fluid Mechanics and Aerospace Sciences* (ed. S. M. Deshpande, A. Prabhu, K. R. Sreenivasan & P. R. Viswanath), pp. 159–193. Interline Publishers.

Sreenivasan, K. R. 1995 On the universality of the Kolmogorov constant. *Phys. Fluids* **7**, 2778.

Tennekes, H. & Lumley, J. L. 1972 *A First Course in Turbulence*. MIT Press.

Vreman, W. A., Sandham, N. D. & Luo, K. H. 1996 Compressible mixing layer growth rate and turbulence characteristics. *J. Fluid Mech.* **320**, 235.

Wilcox, D. C. 1992 Dilatation-dissipation corrections for advanced turbulence models. *AIAA J.* **30**, 2639.

Yeung, P. K. & Zhou, Y. 1997 The universality of the Kolmogorov constant in numerical simulations of turbulence. *Phys. Rev.* E (submitted).

Zank, G .P. & Matthaeus, W. H. 1991 The equations of nearly incompressible fluids: I. Hydrodynamics turbulence and waves. *Phys. Fluids* A **3**, 69.

Zeman, O. 1990 Dilatational dissipation: the concept and application in modeling compressible mixing layers. *Phys. Fluids* A **2**, 178.

Zeman, O. 1991 On the decay of compressible isotropic turbulence. *Phys. Fluids* A **3**, 951.

Zeman, O. & Coleman G. N. 1993 Compressible turbulence subjected to shear and rapid compression. In *Turbulent Shear Flows 8* (ed. F Durst *et al.*). Springer.

Zeman, O. 1993 A new model for super/hypersonic turbulent boundary layers. *AIAA Paper* 93-0897.